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Giant magnon solution and dispersion relation in string theory in $AdS_3 \times S^3 \times T^4$ with mixed flux [☆]

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Abstract

We address the question of the exact form of the dispersion relation for light-cone string excitations in string theory in $AdS_3 \times S^3 \times T^4$ with mixed R–R and NS–NS 3-form fluxes. The analogy with string theory in $AdS_5 \times S^5$ suggests that in addition to the data provided by the perturbative near-BMN expansion and symmetry algebra considerations there is another source of information for the dispersion relation – the semiclassical giant magnon solution. In earlier work in arXiv:1303.1037 and arXiv:1304.4099 we found that the symmetry algebra constraints, which are consistent with a perturbative expansion, do not completely determine the form of the dispersion relation. The aim of the present paper is to fix the dispersion relation by constructing a generalisation of the known dyonic giant magnon soliton on S^3 to the presence of a non-zero NS–NS flux described by a WZ term in the string action (with coefficient q). We find that the angular momentum of this soliton gets shifted by a term linear in world-sheet momentum p. We also discuss the symmetry algebra of the string light-cone S-matrix and show that the exact dispersion relation, which should have the correct perturbative BMN and semiclassical giant magnon limits, should also contain such a linear momentum term. The simplicity of the resulting bound-state picture provides a strong argument in favour of this dispersion relation.

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1. Introduction

Recently two of us studied superstring theory on $AdS_3 \times S^3 \times T^4$ with mixed R–R and NS–NS 3-form fluxes [1,2] with the aim of solving it using the same integrability-based methods as developed in the pure R–R flux case (see [3,4] and references therein). The tree-level light-cone gauge S-matrix for BMN string excitations [5] was computed in [1]. These excitations have the following perturbative dispersion relation

$$\varepsilon_\pm = \sqrt{1 - q^2 + (p \pm q)^2} = \sqrt{(1 \pm qp)^2 + \big(1 - q^2\big)p^2}. \tag{1.1}$$

Here $0 \leqslant q \leqslant 1$ is the coefficient of the NS–NS flux ($\hat{q} = \sqrt{1-q^2}$ is the coefficient of the R–R flux) and p is the spatial momentum of a 2-d string fluctuation.[2] In general, (1.1) is expected to receive corrections at higher orders in the inverse string tension ($\mathrm{h} = \frac{\sqrt{\lambda}}{2\pi}$) expansion. To obtain the exact S-matrix the first step is to find the exact generalisation of the dispersion relation (1.1). Using symmetry considerations, as in the $AdS_5 \times S^5$ case [6], in [2] the exact generalisation of the dispersion relation was suggested to be

$$\varepsilon_\pm = \sqrt{M_\pm^2 + 4\big(1 - q^2\big)\mathrm{h}^2 \sin^2 \frac{\mathrm{p}}{2}}, \tag{1.2}$$

where the "central charge" $M_\pm$ is not uniquely determined. The condition that (1.2) should reduce to (1.1) in the near-BMN limit $\mathrm{h} \gg 1$, $\mathrm{p} \ll 1$ with $p = \mathrm{hp}$ fixed implies that

$$M_\pm = 1 \pm q\mathrm{hp} + \cdots = 1 \pm qp + O\big(\mathrm{h}^{-1}\big). \tag{1.3}$$

If one assumes that the dispersion relation should be manifestly periodic in p (i.e. with $M_\pm$ being a smooth periodic function of p, which would apply if there were an underlying spin chain system) then the simplest consistent form of M would be [2]

$$M_\pm = 1 \pm 2q\mathrm{h} \sin \frac{\mathrm{p}}{2}. \tag{1.4}$$

As was noted in [2], such a manifestly periodic dispersion relation (1.2), (1.4) suggestive of an underlying spin chain picture also naturally emerges upon formally discretizing the spatial direction in the string action (with step h^{-1}).

There is, however, no a priori reason to expect a spin chain interpretation to apply to the string integrable system for $q \neq 0$. It does not apparently apply for $q = 1$ when the world-sheet theory is related to a WZW model (which is solved in conformal gauge using, e.g., an effective free-field representation). For this reason it would be important to have an independent argument for or against the explicitly periodic choice (1.4) made in [2].

In the $AdS_5 \times S^5$ string case, in addition to the light-cone symmetry algebra considerations and the perturbative near-BMN expansion, there was a third string-theory-based source of information for the dispersion relation – the semiclassical giant magnon solution. The aim of the present paper is to use this third approach to complement previous work on the first two approaches [1,2] and shed further light on the exact form of the mixed-flux dispersion relation. Following [7–9], we consider a giant magnon solution on S^3 with two angular momenta (J_1, J_2) and find that its energy is given by $(E, J_1 \to \infty)$

2 The quantized coefficient of the WZ term in the string action is $k = 2\pi q\mathrm{h}$.

$$E - J_1 = \sqrt{M_\pm^2 + 4\left(1-q^2\right)\mathrm{h}^2 \sin^2\frac{\mathrm{p}}{2}}, \tag{1.5}$$

$$M_\pm = J_2 \pm q\mathrm{hp}. \tag{1.6}$$

For $q = 0$ this reduces to the standard dispersion relation for a dyonic giant magnon [8,9]. In the giant magnon construction the momentum p is related to the angle $\Delta\phi_1$ between the end-points of an open rigid string moving along a circle of S^3 so that $\mathrm{p} \in (-\pi, \pi)$. One may formally consider the energy as periodic in p by periodically extending (1.6) to the whole interval $\mathrm{p} \in (-\infty, \infty)$.[3]

The giant magnon solution is interpreted as a bound state of J_2 elementary "magnons" (string excitations) so that for $J_2 = 1$ this relation corresponds to (1.2) with an exact linear expression for $M_\pm$ (i.e. without any higher order corrections in (1.3))

$$M_\pm = 1 \pm q\mathrm{hp}. \tag{1.7}$$

The resulting dispersion relation (1.5), (1.7) has the nice feature that for $q = 1$, i.e. in WZW model limit, it directly reduces to the expected massless dispersion relation

$$q = 1: \quad \varepsilon_\pm = 1 \pm \mathrm{hp}. \tag{1.8}$$

To derive (1.5), (1.6) we shall start with the bosonic string moving in $\mathbb{R} \times S^3$ in the presence of an NS–NS flux, i.e. described by an action with a WZ term proportional to q, and consider its classical solutions (see also [1] and references therein). Some previous discussions of similar classical solutions in this model appeared in [10–12] but they will not be used here. Since the string model on $\mathbb{R} \times S^3$ in the conformal gauge can be interpreted as a principal chiral model with a WZ term proportional to q, to find solutions for $q \neq 0$ from solutions in the $q = 0$ case one may use the fact that the $q \neq 0$ equations of motion written in terms of $SU(2)$ currents are related to the $q = 0$ equations of motion through a world-sheet coordinate transformation.

In Section 2 we will review the classical string equations on $\mathbb{R} \times S^3$ in conformal gauge described by the $SU(2)$ principal chiral model with a WZ term proportional to q. We will then discuss the corresponding conserved charges, pointing out an ambiguity in the action related to boundary terms, and describe a procedure for constructing classical solutions for $q \neq 0$ from their $q = 0$ counterparts, illustrating it on the example of the rigid circular string solution on S^3.

In Section 3 we will construct the dyonic giant magnon solution generalising the solution of [7,8] to the $q \neq 0$ case. We will find the corresponding relation between the energy, the finite angular momentum component J_2, and the effective kink charge, equal to the separation angle $\Delta\phi_1$ between the rigid open string endpoints. Claiming that the latter should be interpreted as in [7,8] as the magnon world-sheet momentum p, we obtain the dispersion relation (1.5), (1.6).

In Section 4 we will further justify this momentum identification by considering the limit of large angular momentum which isolates and effectively decouples fast string motion of extended slowly varying string configurations such as the giant magnon. In this limit the string motion is described by a $q \neq 0$ generalisation of the familiar Landau–Lifshitz model [13–15]. The Landau–Lifshitz equations are known to admit a "spin wave" soliton [16–19] which may be interpreted as the large J_2 limit of the dyonic giant magnon solution. The world-sheet momentum p of this

[3] The periodicity in p becomes irrelevant in the perturbative string theory limit of $\mathrm{h} \gg 1$ when we set $\mathrm{p} = \mathrm{h}^{-1} p$ for fixed p so that p goes to zero.

Landau–Lifshitz soliton has a straightforward definition that confirms its identification with $\Delta\phi_1$ of the giant magnon. The resulting dispersion relation represents the large J_2 limit of (1.5), i.e.

$$E_{\rm LL} = E - J_1 - J_2 = -q\mathrm{hp} + \frac{2(1-q^2)\mathrm{h}^2}{J_2}\sin^2\frac{\mathrm{p}}{2} + O\big(J_2^{-2}\big). \tag{1.9}$$

In Section 5 we will revisit the discussion of the world-sheet S-matrix of the mixed-flux $AdS_3 \times S^3$ theory in [1,2]. We will first review the light-cone symmetry algebra and then suggest a modification to the conjecture for the central charge function $M_\pm$ in [2], switching from (1.7) to (1.4). Doing so, we recover the semiclassical $q \neq 0$ dyonic giant magnon dispersion relation (1.5), (1.6) by considering the bound states of elementary excitations (with J_2 being the number of constituents) and taking an appropriate strong-coupling limit. The simplicity of the bound-state picture provides a strong argument in favour of the linear momentum function (1.7).

Some concluding remarks will be made in Section 6. In Appendix A we will comment on the relation between the dyonic giant magnon solution and the soliton of the corresponding Pohlmeyer reduced theory.

2. Classical string solutions on $\mathbb{R} \times S^3$ for $q \neq 0$

In this section we shall discuss the relation between the $q = 0$ and $q \neq 0$ classical string equations on $\mathbb{R} \times S^3$ that we will use in the following section to find the unique generalisation of the standard $q = 0$ dyonic giant magnon solution of [8] to $q \neq 0$. We will see that the $q = 0$ and $q \neq 0$ equations written in terms of the current $\mathfrak{J} = g^{-1}dg$ are related by a world-sheet coordinate transformation. Our strategy will be (i) to perform this world-sheet coordinate transformation on the $q = 0$ current of a given solution to obtain its $q \neq 0$ counterpart and (ii) starting with this new current to solve for the coordinates of the $q \neq 0$ solution.

The string action in the conformal gauge is equivalent to that of the principal chiral model with a Wess–Zumino term with the coefficient $q \in (0, 1)$

$$S = -\frac{\mathrm{h}}{2}\bigg[\int d^2\sigma\,\frac{1}{2}\mathrm{tr}(\mathfrak{J}_+\mathfrak{J}_-) - q\int d^3\sigma\,\frac{1}{3}\varepsilon^{abc}\mathrm{tr}(\mathfrak{J}_a\mathfrak{J}_b\mathfrak{J}_c)\bigg], \quad \mathfrak{J}_a = g^{-1}\partial_a g, \tag{2.1}$$

where h is the string tension, $g \in SU(2)$ and $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$, $\partial_\pm = \partial_\tau \pm \partial_\sigma$.

2.1. Classical equations

The equation of motion for the above action is

$$(1+q)\partial_-\mathfrak{J}_+ + (1-q)\partial_+\mathfrak{J}_- = 0, \quad \mathfrak{J} = g^{-1}dg, \tag{2.2}$$

or, equivalently

$$(1-q)\partial_-\mathfrak{K}_+ + (1+q)\partial_+\mathfrak{K}_- = 0, \quad \mathfrak{K} = dgg^{-1}. \tag{2.3}$$

Supplemented with the flatness condition (2.2) can be rewritten as

$$\partial_+\mathfrak{J}_- + \frac{1}{2}(1+q)[\mathfrak{J}_+, \mathfrak{J}_-] = 0, \qquad \partial_-\mathfrak{J}_+ - \frac{1}{2}(1-q)[\mathfrak{J}_+, \mathfrak{J}_-] = 0. \tag{2.4}$$

The formal transformation of the world-sheet coordinates

$$\sigma^\pm \to \tilde\sigma^\pm = (1 \pm q)\sigma^\pm \tag{2.5}$$

then maps the $q \neq 0$ current equations to the $q = 0$ equations, provided the currents are left unaltered (i.e. this is not a conformal transformation that leaves the classical equations invariant). Furthermore, the Virasoro conditions (assuming that the target space time coordinate is $t = \kappa\tau$)

$$\mathrm{tr}\big(\mathfrak{J}_\pm^2\big) = -2\kappa^2 \tag{2.6}$$

are invariant under this transformation. Given a solution for $q = 0$ the map (2.5) allows us to construct the $q \neq 0$ counterpart, $\mathfrak{J}$, of the $q = 0$ current. It then remains to solve the defining equations of $\mathfrak{J}$ for the function g, or, e.g., for the 4 real (2 complex) S^3 embedding coordinates X_m, $m = 1, \dots, 4$ (Z_i, $i = 1, 2$)

$$\mathfrak{J}_\pm = g^{-1}\partial_\pm g, \quad g = \begin{pmatrix} Z_1 & Z_2 \\ -Z_2^* & Z_1^* \end{pmatrix} \in SU(2), \tag{2.7}$$

$$Z_1 = X_1 + iX_2, \qquad Z_2 = X_3 + iX_4, \qquad X_m^2 = 1, \qquad |Z_i|^2 = 1. \tag{2.8}$$

The relation $\partial_\pm g = g\mathfrak{J}_\pm$ then gives first order differential equations for the complex embedding coordinates in terms of the current $\mathfrak{J}$

$$\partial_\pm \vec{Z} = \mathfrak{J}_\pm^T \vec{Z}. \tag{2.9}$$

Taking an additional derivative these imply

$$\partial_+\partial_-\vec{Z} = (\partial_-\mathfrak{J}_+)^T\vec{Z} + \mathfrak{J}_+^T\mathfrak{J}_-^T\vec{Z}, \qquad \partial_-\partial_+\vec{Z} = (\partial_+\mathfrak{J}_-)^T\vec{Z} + \mathfrak{J}_-^T\mathfrak{J}_+^T\vec{Z}. \tag{2.10}$$

Subtracting these equations gives the compatibility condition $\partial_+\partial_-\vec{Z} = \partial_-\partial_+\vec{Z}$, which corresponds to the flatness condition for $\mathfrak{J}$. Adding the two equations one obtains

$$\partial_+\partial_-\vec{Z} + \varOmega\vec{Z} + \frac{1}{2}q[\mathfrak{J}_+, \mathfrak{J}_-]^T\vec{Z} = 0, \quad \varOmega = -\frac{1}{2}\mathrm{tr}(\mathfrak{J}_+\mathfrak{J}_-), \tag{2.11}$$

where have we used the fact that since $\mathfrak{J}_\pm$ are traceless and anti-hermitian the anti-commutator $\{\mathfrak{J}_+, \mathfrak{J}_-\}$ is proportional to the identity.

In order to solve for $\vec{Z}$ we decouple (2.9) into two first order equations

$$\mathfrak{J}_-^{21}\partial_+ Z_1 - \mathfrak{J}_+^{21}\partial_- Z_1 + \big(\mathfrak{J}_+^{21}\mathfrak{J}_-^{11} - \mathfrak{J}_-^{21}\mathfrak{J}_+^{11}\big)Z_1 = 0, \tag{2.12}$$

$$\mathfrak{J}_-^{12}\partial_+ Z_2 - \mathfrak{J}_+^{12}\partial_- Z_2 + \big(\mathfrak{J}_+^{12}\mathfrak{J}_-^{22} - \mathfrak{J}_-^{12}\mathfrak{J}_+^{22}\big)Z_2 = 0. \tag{2.13}$$

These linear first order partial differential equations can be solved using the method of characteristics and their solution will involve an undetermined function. At the same time, the original equations (2.9) are four first order equations for two variables, which uniquely determine the solution up to integration constants. Therefore, we still need to impose some additional conditions. This we can do by decoupling the second order equations (2.11) as

$$\partial_+\partial_- Z_1 + q\frac{C_{12}}{\mathfrak{J}_+^{21}}\partial_+ Z_1 + \left(\varOmega + qC_{11} - q\frac{C_{12}}{\mathfrak{J}_+^{21}}\mathfrak{J}_+^{11}\right)Z_1 = 0, \tag{2.14}$$

$$\partial_+\partial_- Z_2 + q\frac{C_{21}}{\mathfrak{J}_+^{12}}\partial_+ Z_2 + \left(\varOmega + qC_{22} - q\frac{C_{21}}{\mathfrak{J}_+^{12}}\mathfrak{J}_+^{22}\right)Z_2 = 0, \tag{2.15}$$

where $C = \frac{1}{2}[\mathfrak{J}_+, \mathfrak{J}_-]^T$. The undetermined function can then be fixed by substituting the solution of the first order equations into the above second order equations.[4]

[4] Notice that Eqs. (2.14) and (2.15) are related through complex conjugation. This does not imply that Z_1 and Z_2 are complex conjugates of each other as we shall see in more detail later on. The above second order differential equations

2.2. Conserved charges

The equations of motion (2.2), (2.3) imply that we have two conserved $SU(2)$ currents, the left-invariant and the right-invariant one

$$L_a = \mathfrak{J}_a - q\epsilon_{ab}\mathfrak{J}^b, \qquad R_a = \mathfrak{K}_a + q\epsilon_{ab}\mathfrak{K}^b, \qquad \partial_a L^a = \partial_a R^a = 0. \tag{2.16}$$

Using these we can construct conserved charges in the standard way

$$Q_L = \mathrm{h}\int \mathrm{d}\sigma\,(\mathfrak{J}_0 + q\mathfrak{J}_1), \qquad Q_R = \mathrm{h}\int \mathrm{d}\sigma\,(\mathfrak{K}_0 - q\mathfrak{K}_1). \tag{2.17}$$

For the dyonic giant magnon solution that is the main subject of this paper, there are a particular pair of charges that we will be interested in

$$\mathrm{J} = -\frac{i}{4}\big(\mathrm{Tr}[Q_L\cdot\sigma_3] + \mathrm{Tr}[Q_R\cdot\sigma_3]\big), \qquad M = -\frac{i}{4}\big(-\mathrm{Tr}[Q_L\cdot\sigma_3] + \mathrm{Tr}[Q_R\cdot\sigma_3]\big). \tag{2.18}$$

We will parametrise the 3-sphere as

$$Z_1 = X_1 + iX_2 = \sin\theta e^{i\phi_1}, \qquad Z_2 = X_3 + iX_4 = \cos\theta e^{i\phi_2}, \tag{2.19}$$

or, equivalently, in terms of the Euler angles

$$g = e^{\frac{i}{2}\theta_L\sigma_3}e^{\frac{i}{2}\psi\sigma_2}e^{\frac{i}{2}\theta_R\sigma_3}, \quad \psi = \pi - 2\theta,\ \theta_L = \phi_1 + \phi_2,\ \theta_R = \phi_1 - \phi_2. \tag{2.20}$$

The bosonic string action (2.1) then takes the form

$$\begin{aligned} S = -\frac{\mathrm{h}}{2}\int \mathrm{d}^2\sigma\,\big[&\partial_a\theta\partial^a\theta + \sin^2\theta\partial_a\phi_1\partial^a\phi_1 + \cos^2\theta\partial_a\phi_2\partial^a\phi_2 \\ &+ q(\cos 2\theta + \mathrm{c})(\dot\phi_1\acute\phi_2 - \dot\phi_2\acute\phi_1)\big], \end{aligned} \tag{2.21}$$

where $a, b = 0, 1$ stand for the world-sheet coordinates τ, σ with the metric $\eta = \mathrm{diag}(-1, 1)$ and $\dot{} = \partial_\tau$, $\acute{} = \partial_\sigma$. The last q-dependent term comes from the Wess–Zumino term, in which the parameter c corresponds to an ambiguity in defining a local 2-d action.[5] The c-term is a total derivative and does not, of course, affect the equations of motion. However, if we consider string solutions with non-trivial boundary conditions (which includes the case of interest – the dyonic giant magnon) then it will affect the corresponding Noether global charges as we shall discuss below.

As we are using the conformal gauge with the residual conformal symmetry fixed by choosing $t = \kappa\tau$ the Virasoro constraints take the following explicit form

$$\begin{aligned} &\dot\theta^2 + \acute\theta^2 + \sin^2\theta\big(\dot\phi_1^2 + \acute\phi_1^2\big) + \cos^2\theta\big(\dot\phi_2^2 + \acute\phi_2^2\big) = \kappa^2, \\ &\dot\theta\acute\theta + \sin^2\theta\dot\phi_1\acute\phi_1 + \cos^2\theta\dot\phi_2\acute\phi_2 = 0. \end{aligned} \tag{2.22}$$

admit two separate solutions corresponding to roots of a quadratic equation. Requiring that the final solution is consistent with the original $q = 0$ solution then uniquely fixes the choice of these roots giving rise to solutions for Z_1 and Z_2 that in general are not related by complex conjugation.

[5] In general, the string couples locally to the antisymmetric B-field, while the defining equations – the conformal invariance conditions or, to leading order, the supergravity equations of motion – depend on the three-form field strength H. Therefore, there is a gauge freedom in choice of the B-field and in the presence of the boundary this necessitates a boundary term parametrising this ambiguity. At the moment we will leave it arbitrary and fix it later via natural physical requirements appropriate for the giant magnon solution.

The translational invariance of the full string action under shifts of t, ϕ_1 and ϕ_2 leads to the following conserved Noether charges: the energy and the angular momenta (here $\sigma \in (-\pi, \pi)$)

$$E = 2\pi \mathrm{h}\kappa, \tag{2.23}$$

$$J_1 = \mathrm{h} \int d\sigma \left[\sin^2\theta \dot{\phi}_1 - \frac{q}{2}(\cos 2\theta + \mathrm{c})\acute{\phi}_2 \right], \tag{2.24}$$

$$J_2 = \mathrm{h} \int d\sigma \left[\cos^2\theta \dot{\phi}_2 + \frac{q}{2}(\cos 2\theta + \mathrm{c})\acute{\phi}_1 \right]. \tag{2.25}$$

Here $J_{1,2}$ follow directly from the action (2.21). The c-terms are of course total derivatives and thus contribute only if ϕ_1 or ϕ_2 have non-trivial boundary values or if periodicity in σ is not imposed (as will be the case for the giant magnon solution).

Let us compare (2.24), (2.25) with the charges J and M (2.18) that were derived from the $SU(2)$-invariant currents (2.17). Substituting the parametrisation (2.19) into (2.17) we find

$$\mathrm{J} = \mathrm{h} \int d\sigma \left[\sin^2\theta \dot{\phi}_1 - \frac{q}{2}(\cos 2\theta + 1)\acute{\phi}_2 \right], \tag{2.26}$$

$$M = \mathrm{h} \int d\sigma \left[\cos^2\theta \dot{\phi}_2 + \frac{q}{2}(\cos 2\theta - 1)\acute{\phi}_1 \right], \tag{2.27}$$

and hence

$$J_1 = \mathrm{J} - \frac{1}{2}\mathrm{h}q(\mathrm{c} - 1)\Delta\phi_2, \qquad J_2 = M - \frac{1}{2}\mathrm{h}q(\mathrm{c} + 1)\Delta\phi_1, \tag{2.28}$$

$$\Delta\phi_i = \phi_i(\pi) - \phi_i(-\pi). \tag{2.29}$$

Thus to match J and M with J_1 and J_2 we need different choices of c $(= \pm 1)$, i.e. J and M cannot be obtained as Noether charges from a local action (2.21) with equations of motion equivalent to (2.2). This, of course, is not a contradiction as the difference appears only due to the boundary "twist" terms $\Delta\phi_i$, but if non-zero such terms break manifest $SU(2)$ symmetry.[6]

The dyonic giant magnon solution we will be interested in is a classical soliton representing a "bound state" of string excitations above the BMN vacuum. The latter corresponds to a point-like string moving along a great circle of S^3

$$\theta = \frac{\pi}{2}, \qquad \phi_1 = \kappa\tau, \qquad \phi_2 = 0. \tag{2.30}$$

For the point-like BMN solution

$$E - J_1 = 0, \quad J_1 = \mathrm{J}. \tag{2.31}$$

In the $q = 0$ case, the giant magnon limit [7] involves taking both E and J_1 to infinity (i.e. $\kappa \to \infty$) with their difference held fixed

$$E, \quad J_1 \to \infty, \qquad \epsilon \equiv E - J_1, \quad J_2 = \text{fixed}. \tag{2.32}$$

Also, as in [7] the string is assumed to be open so that rescaling τ and σ by $\kappa \to \infty$ the spatial interval may be decompactified

$$x = \kappa\sigma, \quad \kappa \to \infty, \ x \in (-\infty, +\infty), \tag{2.33}$$

[6] Let us note also that in general, the currents conserved ($\partial_a j_i^a = 0$) on the equations of motion are defined modulo a trivial term $\epsilon^{ab}\partial_b f_i$ where the functions f_i (that may, in principle, break some manifest global symmetries) contribute to the corresponding charges only if they have non-trivial boundary twists.

and the non-zero angle between the end points of the string

$$\Delta\phi_1 = \phi_1(x=\infty) - \phi_1(x=-\infty), \tag{2.34}$$

may be related to the 2-d momentum p. Then ϵ, which plays the role of the energy of the state relative to the BMN vacuum, can be expressed as a function of p and J_2.

As we shall see below, for $q \neq 0$ the requirement that $E - J_1$ remains finite in the $\kappa \to \infty$ limit (and also the classical action is finite when evaluated on one period of the dyonic giant magnon solution) implies that

$$\mathrm{c} = 1. \tag{2.35}$$

In this case the action (2.21) becomes explicitly

$$S = -\frac{\mathrm{h}}{2}\int d^2\sigma\,\big(\partial_a\theta\partial^a\theta + \sin^2\theta\,\partial_a\phi_1\partial^a\phi_1 + \cos^2\theta\,\partial_a\phi_2\partial^a\phi_2 + 2q\epsilon^{ab}\cos^2\theta\,\partial_a\phi_1\partial_b\phi_2\big). \tag{2.36}$$

The physical reason for this particular choice of B_{mn}-term in the string action is that it vanishes at $\theta = \frac{\pi}{2}$. This implies the vanishing of force on the ends of the open string (representing the giant magnon solution) moving along the great circle corresponding to ϕ_1. As usual, the boundary term in the variation of the string action specifies the boundary conditions for the open string end-points. The variation of (2.36) under the variation of ϕ_1 gives the condition $\int d\tau\,\delta\phi_1(\sin^2\theta\,\partial_\sigma\phi_1 - q\cos^2\theta\,\partial_\tau\phi_2)|_{\sigma=0,\pi} = 0$. Since the end-points of the giant magnon move along the great circle $\theta|_{\sigma=0,\pi} = \frac{\pi}{2}$ the q-dependent term vanishes and we just have the standard free-ends condition $\partial_\sigma\phi_1|_{\sigma=0,\pi} = 0$.

From (2.29) we then get

$$\mathrm{J} = J_1, \qquad M = J_2 - q\mathrm{h}\Delta\phi_1. \tag{2.37}$$

Here $\Delta\phi_1$ plays the role of kink charge, which, as for $q = 0$, can be identified with the 2-d spatial momentum p of the soliton. Recalling that the quantized WZ level k is related to q as $k = 2\pi\mathrm{h}q$, we may write M as

$$M = J_2 - q\mathrm{hp} = J_2 - k\frac{\mathrm{p}}{2\pi}, \qquad \mathrm{p} = \Delta\phi_1. \tag{2.38}$$

Here $\Delta\phi_1 \in (0,\pi)$ but being an angular coordinate it may defined modulo 2π, and then the same may be assumed about p, i.e. M may be considered as a periodic function of p.[7]

Also, for a physical closed string $\Delta\phi_1$ should be equal to $2\pi n$ where n is an integer winding number, or, equivalently, the total momentum of a bound state of magnons representing a physical state should be quantized

$$\sum_i \mathrm{p}_i = 2\pi n. \tag{2.39}$$

This is consistent with both M and J_2 in (2.38) taking integer values for such states.

This relation between M and J_2 is suggestive of how the dyonic giant magnon dispersion relation is to be modified in the presence of the NS–NS flux (cf. (1.5), (1.6)).

[7] The issue of periodicity is a subtle one and we will return to it later in Section 3.3 after we have derived the relevant expressions for the energy and angular momenta of the dyonic giant magnon as functions of the solution parameters (which include $\Delta\phi_1$).

2.3. An example of a solution: rigid circular string

Before turning to the construction of the giant magnon solution for $q \neq 0$ let us illustrate the general procedure of finding $q \neq 0$ solutions from their $q = 0$ counterparts on the example of a rigid circular string on S^3 [20,21]. The standard $q = 0$ solution written in the embedding coordinates reads

$$Z_1 = \frac{1}{\sqrt{2}} \exp\big[i(\omega + m)\sigma^+ + i(\omega - m)\sigma^-\big],$$
$$Z_2 = \frac{1}{\sqrt{2}} \exp\big[i(\omega - m)\sigma^+ + i(\omega + m)\sigma^-\big], \quad m^2 + \omega^2 = \kappa^2. \tag{2.40}$$

For this solution the $SU(2)$ currents are (see (2.7))

$$\mathfrak{J}_+ = i \begin{pmatrix} m & \omega \exp[-2im(\sigma^+ - \sigma^-)] \\ \omega \exp[2im(\sigma^+ - \sigma^-)] & -m \end{pmatrix},$$
$$\mathfrak{J}_- = i \begin{pmatrix} -m & \omega \exp[-2im(\sigma^+ - \sigma^-)] \\ \omega \exp[2im(\sigma^+ - \sigma^-)] & m \end{pmatrix},$$
$$\Omega = -\frac{1}{2}\mathrm{tr}(\mathfrak{J}_+\mathfrak{J}_-) = \omega^2 - m^2. \tag{2.41}$$

Performing the world-sheet coordinate transformation (2.5) gives the $q \neq 0$ currents

$$\mathfrak{J}_+ = i \begin{pmatrix} m & \omega \exp[-2im[\sigma^+ - \sigma^- + q(\sigma^+ + \sigma^-)]] \\ \omega \exp[2im[\sigma^+ - \sigma^- + q(\sigma^+ + \sigma^-)]] & -m \end{pmatrix},$$
$$\mathfrak{J}_- = i \begin{pmatrix} -m & \omega \exp[-2im[\sigma^+ - \sigma^- + q(\sigma^+ + \sigma^-)]] \\ \omega \exp[2im[\sigma^+ - \sigma^- + q(\sigma^+ + \sigma^-)]] & m \end{pmatrix},$$
$$C = 2m\omega \begin{pmatrix} 0 & \exp[2im(\sigma^+ - \sigma^- + q(\sigma^+ + \sigma^-))] \\ -\exp[-2im(\sigma^+ - \sigma^- + q(\sigma^+ + \sigma^-))] & 0 \end{pmatrix}, \tag{2.42}$$

with Ω unchanged, while the decoupled equations for the embedding coordinates (2.14)–(2.15) become

$$\partial_+\partial_- Z_1 - 2qmi\partial_+ Z_1 + \big(\omega^2 - m^2 - 2qm^2\big) Z_1 = 0, \tag{2.43}$$
$$\partial_+\partial_- Z_2 + 2qmi\partial_+ Z_2 + \big(\omega^2 - m^2 - 2qm^2\big) Z_2 = 0. \tag{2.44}$$

Fourier decomposing the solution as

$$Z_1 = a_n \exp[i\mu_n\tau + in\sigma], \qquad Z_2 = b_n \exp[i\nu_n\tau - in\sigma], \tag{2.45}$$

and requiring that the modes reduce to those of the $q = 0$ circular string solution one obtains

$$\mu_m = qm + \sqrt{q^2m^2 + \omega^2}, \qquad \nu_m = -qm + \sqrt{q^2m^2 + \omega^2}. \tag{2.46}$$

The normalisation condition $|Z_1|^2 + |Z_2|^2 = 1$ together with the Virasoro constraints $\partial_\pm Z_1 \times \partial_\pm Z_1^* + \partial_\pm Z_2 \partial_\pm Z_2^* = \kappa^2 = \omega^2 + m^2$ then determine a_n and b_n up to a phase giving

$$Z_1 = \sqrt{\frac{W - qm}{2W}} \exp\big(i\big[(W + qm)\tau + m\sigma\big]\big), \tag{2.47}$$
$$Z_2 = \sqrt{\frac{W + qm}{2W}} \exp\big(i\big[(W - qm)\tau - m\sigma\big]\big), \quad W = \sqrt{\omega^2 + q^2m^2}. \tag{2.48}$$

In the parametrisation (2.19) this solution takes the form

$$\sin\theta = \sqrt{\frac{W-qm}{2W}}, \qquad \phi_1 = (W+qm)\tau + m\sigma, \qquad \phi_2 = (W-qm)\tau - m\sigma. \tag{2.49}$$

The two angular momenta associated to shifts in ϕ_1 and ϕ_2 computed from (2.24), (2.25) are

$$J_1 = J_2 = \pi \mathrm{h}(W + \mathrm{c}qm), \tag{2.50}$$

where c parametrises the ambiguity in the choice of the total derivative term in the action (2.21). The expression for energy then takes the form

$$E = 2\pi \mathrm{h}\kappa = \sqrt{(J - 2\pi \mathrm{hc}qm)^2 + 4\pi^2\mathrm{h}^2 m^2\big(1-q^2\big)}, \quad J \equiv J_1 + J_2 = 2J_1. \tag{2.51}$$

Expanding in large J we get

$$E = J - 2\pi \mathrm{hc}qm + \frac{2\pi^2\mathrm{h}^2(1-q^2)m^2}{J} + O\big(J^{-2}\big). \tag{2.52}$$

The choice $\mathrm{c} = 0$ here gives the standard BMN limit $E = J$ when $J \to \infty$.

3. Dyonic giant magnon on $\mathbb{R} \times S^3$ in the presence of NS–NS flux

3.1. Review of $q = 0$ case

Let us start with a review of the standard dyonic giant magnon solution on S^3 in the absence of an NS–NS flux, i.e. for $q = 0$ in the action (2.21). In the notation of Section 2 the dyonic giant magnon solution, labelled by the two independent parameters b (or v) and ρ, takes the form [8]

$$Z_1 = \frac{[b + i\tanh(\mathcal{X}\cos\rho)]\exp(it)}{(1+b^2)^{1/2}}, \qquad Z_2 = \frac{\operatorname{sech}(\mathcal{X}\cos\rho)\exp(i\mathcal{T}\sin\rho)}{(1+b^2)^{1/2}}, \tag{3.1}$$

$$b = \frac{v\sec\rho}{\sqrt{1-v^2}}, \quad v \in (0,1), \ \rho \in \left(0, \frac{\pi}{2}\right), \ b \in (0,\infty), \tag{3.2}$$

where $\mathcal{X}$ and $\mathcal{T}$ are related to the world-sheet coordinates τ, σ through a boost of velocity v and a rescaling by κ

$$\mathcal{X} = \frac{x - vt}{\sqrt{1-v^2}}, \qquad \mathcal{T} = \frac{t - vx}{\sqrt{1-v^2}}, \tag{3.3}$$

$$t = \kappa\tau, \qquad x = \kappa\sigma, \qquad \tau \in (-\infty,\infty), \qquad \sigma \in (-\pi,\pi), \qquad x \in (-\infty,\infty). \tag{3.4}$$

Here we have already taken the limit $\kappa \to \infty$ and thus "decompactified" the spatial direction x. $x \to \pm\infty$ correspond to the endpoints of the string moving in the ϕ_1 direction, while $\rho \in (0, \frac{\pi}{2})$ is the parameter associated with the angular momentum in the ϕ_2 direction. We may of course extend the parameter ranges so that $v \in (-1,1)$, $\rho \in (-\pi,\pi) \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ to cover also the soliton moving in the opposite direction. These ranges correspond to $b \in (-\infty,\infty)$ and hence $\Delta\phi_1 \in (-\pi,\pi)$.

This solution satisfies the boundary conditions

$$x \to \pm\infty: \quad Z_1 \to \exp\left(it \pm i\frac{\Delta\phi_1}{2}\right), \quad Z_2 \to 0, \tag{3.5}$$

where

$$\Delta\phi_1 = 2\arctan b^{-1} \in (0, \pi) \tag{3.6}$$

corresponds to the angle between the rigid open string endpoints which move in the ϕ_1 direction on the great circle $\theta = \frac{\pi}{2}$.

The finite combination of energy E with J_1 and the angular momentum J_2 for this solution are given by

$$E - J_1 = \frac{2\mathrm{h}}{1+b^2}\frac{(1+b^2\cos^2\rho)^{1/2}}{\cos\rho}, \qquad J_2 = \frac{2\mathrm{h}}{1+b^2}\tan\rho. \tag{3.7}$$

The case of $\rho = 0$ thus corresponds to the S^2 giant magnon of [7] (then Z_2 in (3.1) becomes real) with $J_2 = 0$. In addition to J_2 another "charge" parameter of this solution is the "kink charge" $\Delta\phi_1$. Expressing the energy in terms of these charges we get

$$E - J_1 = \sqrt{J_2^2 + \frac{4\mathrm{h}^2}{1+b^2}} = \sqrt{J_2^2 + 4\mathrm{h}^2\sin^2\frac{\Delta\phi_1}{2}}. \tag{3.8}$$

This becomes the usual dyonic giant magnon dispersion relation upon the identification [7,8] of the magnon momentum with the separation angle: $\mathrm{p} = \Delta\phi_1$.

Let us mention that if one considers a more general solution where the string moves also along an S^1 in the torus part of $AdS_3 \times S^3 \times T^4$ then the dispersion relation (3.8) is modified as follows:

$$\sqrt{E^2 - P^2} - J_1 = \sqrt{J_2^2 + 4\mathrm{h}^2\sin^2\frac{\mathrm{p}}{2}}, \tag{3.9}$$

where P is the (large) momentum in S^1 with E, P and J_1 scaling as $\kappa \to \infty$. This follows simply from the formal Lorentz invariance in the $\mathbb{R}_t \times S^1_\psi$ subspace in the decompactification limit (equivalently, the contribution of the $\mathbb{R}_t$ and S^1_ψ to the Virasoro condition can be absorbed into a rescaling of τ and σ).

3.2. *Dyonic giant magnon for $q \neq 0$*

Let us now generalise the above solution to the $q \neq 0$ case using the procedure outlined in Section 2. First we are to re-express the current, constructed from the $q = 0$ solution (3.1) via (2.7), in terms of $\tilde\sigma^\pm$, defined in (2.5), giving us the current $\mathfrak{J}$ of the $q \neq 0$ solution. Anticipating that the $q \neq 0$ solution is again most conveniently written in terms of boosted world-sheet coordinates we introduce the boosted world-sheet coordinates $\tilde{\mathcal{X}}$, $\tilde{\mathcal{T}}$ which are related to the $q \neq 0$ world-sheet coordinates $\tilde t, \tilde x$ by a boost of velocity v, i.e.

$$\begin{aligned} &\tilde\sigma^\pm = (1 \pm q)\sigma^\pm = \frac{1}{2}(\tilde\tau \pm \tilde\sigma), \qquad \tilde t = \kappa\tilde\tau, \qquad \tilde x = \kappa\tilde\sigma, \\ &\tilde{\mathcal{X}} = \frac{\tilde x - v\tilde t}{\sqrt{1-v^2}}, \qquad \tilde{\mathcal{T}} = \frac{\tilde t - v\tilde x}{\sqrt{1-v^2}}. \end{aligned} \tag{3.10}$$

Note that the boosted world-sheet coordinates of the $q = 0$ and $q \neq 0$ cases are related via

$$\tilde{\mathcal{X}} = \mathcal{X} + q\mathcal{T}, \qquad \tilde{\mathcal{T}} = \mathcal{T} + q\mathcal{X}. \tag{3.11}$$

It is useful also to introduce rescaled coordinates $\tilde\xi$ and $\tilde\eta$, defined as

$$\tilde\xi = \tilde{\mathcal{X}}\cos\rho, \qquad \tilde\eta = \tilde{\mathcal{T}}\sin\rho. \tag{3.12}$$

The coordinate transformation from the original light-cone coordinates $\sigma^{\pm}$ to $\tilde{\xi}$, $\tilde{\eta}$ is then

$$\sigma^{+}=\frac{1}{2\kappa(1+q)}\sqrt{\frac{1+v}{1-v}}\left(\frac{\tilde{\eta}}{\sin\rho}+\frac{\tilde{\xi}}{\cos\rho}\right),$$
$$\sigma^{-}=\frac{1}{2\kappa(1-q)}\sqrt{\frac{1-v}{1+v}}\left(\frac{\tilde{\eta}}{\sin\rho}-\frac{\tilde{\xi}}{\cos\rho}\right). \tag{3.13}$$

Written in $\tilde{\xi}$, $\tilde{\eta}$ coordinates Eqs. (2.12)–(2.13) become

$$\partial_{\tilde{\xi}}Z_1+A\partial_{\tilde{\eta}}Z_1+BZ_1=0,\qquad \partial_{\tilde{\xi}}Z_2+A^{*}\partial_{\tilde{\eta}}Z_2+B^{*}Z_2=0, \tag{3.14}$$

where

$$A=\tan\rho\frac{(1+q)\sqrt{\frac{1-v}{1+v}}\mathfrak{J}^{21}_{-}-(1-q)\sqrt{\frac{1+v}{1-v}}\mathfrak{J}^{21}_{+}}{(1+q)\sqrt{\frac{1-v}{1+v}}\mathfrak{J}^{21}_{-}+(1-q)\sqrt{\frac{1+v}{1-v}}\mathfrak{J}^{21}_{+}}=\tan\rho\frac{m+qb+iq\tanh\tilde{\xi}}{qm+b+i\tanh\tilde{\xi}}, \tag{3.15}$$

$$B=\frac{\kappa^{-1}\cos^{-1}\rho(\mathfrak{J}^{21}_{+}\mathfrak{J}^{11}_{-}-\mathfrak{J}^{21}_{-}\mathfrak{J}^{11}_{+})}{(1+q)\sqrt{\frac{1-v}{1+v}}\mathfrak{J}^{21}_{-}+(1-q)\sqrt{\frac{1+v}{1-v}}\mathfrak{J}^{21}_{+}}$$
$$=-i\frac{\operatorname{sech}^2\xi+(k+i\tanh\tilde{\xi})(b+iu\tanh\tilde{\xi})}{(u+1)(b+qm+i\tanh\tilde{\xi})}, \tag{3.16}$$

$$u=\frac{\sqrt{1+b^2\cos^2\rho}}{\sin\rho},\qquad m=(1-u)\tan\rho. \tag{3.17}$$

We can then write (3.14) as an ordinary differential equation

$$\frac{\mathrm{d}Z_1}{\mathrm{d}\tilde{\xi}}+BZ_1=0 \tag{3.18}$$

valid along the characteristic curve

$$\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\tilde{\xi}}=A(\tilde{\xi})\quad\Rightarrow\quad \tilde{\eta}=\int\mathrm{d}\tilde{\xi}\,A(\tilde{\xi})+C_0. \tag{3.19}$$

Evaluating the integrals of A and B we obtain

$$I_1=\int\mathrm{d}\tilde{\xi}\,B(\tilde{\xi})=-\frac{2i[k+k^2s_1+(k-s_1)u]\tilde{\xi}}{2(1+s_1^2)(1+u)}$$
$$+\ln\cosh\tilde{\xi}-\frac{[1+k^2+s_1(s_1-k)(1+u)]\ln[2(s_1\cosh\tilde{\xi}+i\sinh\tilde{\xi})^2]}{2(1+s_1^2)(1+u)},$$
$$I_2=\int\mathrm{d}\tilde{\xi}\,A(\tilde{\xi})=\tan\rho\frac{2(1+s_1s_2)\tilde{\xi}-im(1-q^2)\ln[2(s_1\cosh\tilde{\xi}+i\sinh\tilde{\xi})^2]}{2(1+s_1^2)},$$
$$s_1=k+qm,\ s_2=m+qk. \tag{3.20}$$

The solution for Z_1 is then obtained by integrating (3.18)

$$Z_1=f\big(C_0(\tilde{\xi},\tilde{\eta})\big)\exp\left[-\int\mathrm{d}\tilde{\xi}\,B(\tilde{\xi})\right]=f\big(\tilde{\eta}-I_2(\tilde{\xi})\big)\exp\big[-I_1(\tilde{\xi})\big]. \tag{3.21}$$

We can determine f by substituting this solution into (2.14). This gives

$$f''(x) - 2rf'(x) + r^2 - \delta^2 = 0, \tag{3.22}$$

$$r \equiv \frac{i}{2}\left(\frac{1}{\sin\rho\sqrt{1-v^2}} - 1\right), \qquad \delta \equiv \frac{i}{2}\left(1 + \frac{1+q(q-2v)}{\sin\rho(1-q^2)\sqrt{1-v^2}}\right), \tag{3.23}$$

which has the solutions

$$f(x) = e^{a_\pm x}, \quad a_\pm = r \pm \delta. \tag{3.24}$$

Requiring that in the limit $q \to 0$ we recover the dyonic giant magnon solution (3.1) leads to

$$f(z) = e^{a_+ z} = \exp\left(i\frac{z}{\sin\rho\sqrt{1-v^2}}\frac{1-qv}{1-q^2}\right). \tag{3.25}$$

We can now determine the Z_2 solution by taking the complex conjugate of (3.21), but to ensure the correct $q = 0$ limit in this case we should take

$$f(z) = e^{a_- z} = \exp\left(iz\left[\frac{q(q-v)}{\sin\rho(1-q^2)\sqrt{1-v^2}} + 1\right]\right). \tag{3.26}$$

After fixing the normalisation constants using the Virasoro condition and $|Z_1|^2 + |Z_1|^2 = 1$ we obtain the solution written in terms of the original $\mathcal{X}$, $\mathcal{T}$ coordinates (3.3)

$$Z_1 = \frac{(\tilde{b} + i\tanh[\cos\rho(\mathcal{X} + q\mathcal{T})])\exp(it)}{(1+\tilde{b}^2)^{1/2}}, \tag{3.27}$$

$$Z_2 = \frac{\operatorname{sech}[\cos\rho(\mathcal{X} + q\mathcal{T})]\exp(i[\sin\rho(\mathcal{T} + q\mathcal{X}) - qx])}{(1+\tilde{b}^2)^{1/2}}, \tag{3.28}$$

$$\tilde{b} = \sec\rho\left(\frac{v-q}{\sqrt{1-v^2}} + q\sin\rho\right). \tag{3.29}$$

This generalises (3.1), (3.2) to the $q \neq 0$ case. It is straightforward to verify that the solution (3.27)–(3.29) satisfies the defining equations (2.9). Written in the parametrisation (2.19) it takes the form[8]

$$\cos\theta = \frac{\operatorname{sech}[\cos\rho(\mathcal{X} + q\mathcal{T})]}{(1+\tilde{b}^2)^{1/2}}, \tag{3.30}$$

$$\phi_1 = t + \arctan\left(\tilde{b}^{-1}\tanh\left[\cos\rho(\mathcal{X} + q\mathcal{T})\right]\right), \qquad \phi_2 = \sin\rho(\mathcal{T} + q\mathcal{X}) - qx, \tag{3.31}$$

where as in (3.3) here $\mathcal{X} = \frac{x-vt}{\sqrt{1-v^2}}$, $\mathcal{T} = \frac{t-vx}{\sqrt{1-v^2}}$.

The asymptotics of this $q \neq 0$ dyonic giant magnon solution (3.27), (3.28) have the same form as in the $q = 0$ case (3.5)

$$x \to \pm\infty: \quad Z_1 \to \exp\left(it \pm i\frac{\Delta\phi_1}{2}\right), \qquad Z_2 \to 0, \tag{3.32}$$

$$\Delta\phi_1 = 2\arctan\tilde{b}^{-1} \in (0, \pi). \tag{3.33}$$

Here we have restricted so that $\Delta\phi_1 \in (0, \pi)$, corresponding to $\tilde{b} \in (0, \infty)$. As in the $q = 0$ case these ranges can be extended to $(-\pi, \pi)$ and $(-\infty, \infty)$ respectively.

[8] One can check that this solution remains valid also for $q = 1$ (it satisfies the Virasoro constraints and equations of motion for (2.21)) even though the world-sheet coordinate transformation (2.5) which we used to derive it becomes degenerate. Furthermore, written in terms of the group element (2.20) the solution factorises as expected: $g = \exp(\frac{i}{2}(t - x)\sigma_3) \cdot g_R(t + x)$. It is interesting to note that the right-invariant current is particularly simple in this limit: $\partial_- g g^{-1} = i\sigma_3$.

3.3. Conserved charges and dispersion relation

For the $q=0$ dyonic giant magnon the energy E and the angular momentum J_1 diverge with their difference staying finite. This is no longer true in general for $q \neq 0$: the behaviour of $E-J_1$ happens to depend on the definition of J_1 in (2.24) which is sensitive to the total derivative ambiguity ($\sim \mathrm{c}$) in the Wess–Zumino term in (2.21).[9] We find from (2.23)–(2.25)

$$E-J_1=\mathrm{h}\int_{-\infty}^{\infty} \mathrm{d}x \left(1-\left[\sin^2\theta\,\partial_t\phi_1-\frac{q}{2}(\cos 2\theta+\mathrm{c})\partial_x\phi_2\right]\right), \tag{3.34}$$

$$J_2=\mathrm{h}\int_{-\infty}^{\infty} \mathrm{d}x \left[\cos^2\theta\,\partial_t\phi_2+\frac{q}{2}(\cos 2\theta+\mathrm{c})\partial_x\phi_1\right], \tag{3.35}$$

where we used the rescaled world-sheet coordinates $(t,x)=(\kappa\tau,\kappa\sigma)$ with $t,x\in(-\infty,\infty)$. Computing these integrals for the solution (3.30), (3.31) we find

$$E-J_1=2\mathrm{h}\frac{\sqrt{1-q^2+(\tilde{b}\cos\rho-q\sin\rho)^2}}{(1+\tilde{b}^2)\cos\rho}+\frac{1}{2}\mathrm{h}q(\mathrm{c}-1)\Delta\phi_2, \tag{3.36}$$

$$\Delta\phi_2=-\cos\rho(q\cos\rho+\tilde{b}\sin\rho)x|_{-\infty}^{\infty}, \tag{3.37}$$

$$J_2=M+\frac{1}{2}(\mathrm{c}+1)\mathrm{h}q\Delta\phi_1,\quad M=2\mathrm{h}\frac{\tan\rho-q\tilde{b}}{1+\tilde{b}^2}, \tag{3.38}$$

where $\Delta\phi_1$ is given in (3.33), the divergent expression for $\Delta\phi_2=\phi_2(x=\infty)-\phi_2(x=-\infty)$ follows from (3.31) and M was defined in (2.18), (2.27). We conclude that $E-J_1$ is finite only if $\mathrm{c}=1$.[10] Remarkably, this is exactly the case (cf. (2.28)) when the charge J_1 (2.24) coincides with J in (2.18), (2.26) which corresponds to manifestly $SU(2)$ invariant current.

Eliminating ρ and expressing $\tilde{b}$ in terms of $\Delta\phi_1$ in (3.33) gives

$$\mathrm{c}=1:\quad E-J_1=\sqrt{M^2+4\mathrm{h}^2\left(1-q^2\right)\sin^2\frac{\Delta\phi_1}{2}},\quad M=J_2-q\mathrm{h}\Delta\phi_1. \tag{3.39}$$

Let us comment on the values of parameters here (with $q\in(0,1)$). As in the $q=0$ case, when constructing the solution we restrict to $\Delta\phi_1\in(0,\pi)$, or equivalently $\tilde{b}\in(0,\infty)$. Taking also $\rho\in(0,\frac{\pi}{2})$, this implies the restriction $v>v_*(q,\rho)>v_*(q,0)=q$, where v_* is a function of q and ρ whose explicit form follows from (3.29). As before, we may extend the parameter ranges so that $v\in(-1,1)$ and $\rho\in(-\pi,\pi)\setminus\{-\frac{\pi}{2},\frac{\pi}{2}\}$ and thus $\tilde{b}\in(-\infty,\infty)$ and $\Delta\phi_1\in(-\pi,\pi)$.

Note also that M in (3.38) is single-valued; $J_2=M+q\mathrm{h}\Delta\phi_1$ formally shifts if we shift $\Delta\phi_1$ by its period. As was already mentioned in Section 2.2, the shift is integer as the WZ level $k=2\pi\mathrm{h}q$ should be quantized.

[9] The infinite contribution of the total derivative term comes from the infinite (in the $\kappa\to\infty$ limit) number of "windings" of the $q\neq 0$ giant magnon around the circle of ϕ_2, i.e. $\Delta\phi_2=\infty$ in (3.31).

[10] Let us note again that this choice is not related to the parameters of the solution itself but only to the total derivative term in the action (2.21) or to the definition of the corresponding Noether charge J_1. Let us mention also that the importance of similar WZ-term related boundary terms in the presence of non-trivial kinks was emphasised in a similar context in [22].

It remains to relate the "kink charge" $\Delta\phi_1$ (3.33) to the world-sheet momentum p. In general, there is no universal definition of world-sheet momentum (the total momentum vanishes as we are dealing with a reparametrisation-invariant theory). In the present case the preferred gauge used to define the near-BMN S-matrix is the uniform light-cone gauge (see [23] for a review). In the $q=0$ case the momentum was identified in [7,8] with the angular separation

$$\mathrm{p}=\Delta\phi_1, \tag{3.40}$$

and this relation was indeed demonstrated to apply in the uniform light-cone gauge for the original ($J_2=0$) [7] giant magnon [24,23]. Heuristically, the relation (3.40) is not expected to change upon switching on non-zero values of J_2 and q. First, there should be no momentum flow in ϕ_2 direction in (3.31) as it is linear in t and x, i.e. the relevant momentum should be associated with ϕ_1. Expressing ϕ_1 and ϕ_2 in terms of two other string coordinates – t and θ in (3.30), (3.31) and treating θ as a spatial coordinate along the string ($\cos\theta$ changes from its maximal value to zero and then back) we get[11]

$$\phi_1(t,\theta)=t+\arctan\Big[\tilde{b}^{-1}\sqrt{1-\big(1+\tilde{b}^2\big)\cos^2\theta}\Big], \tag{3.41}$$

$$\phi_2(t,\theta)=wt+r\,\mathrm{arccosh}\Big[\Big(\sqrt{1+\tilde{b}^2}\cos\theta\Big)^{-1}\Big], \tag{3.42}$$

$$w=\frac{(1-q^2)\sin\rho\sqrt{1-v^2}-q(v-q)}{1-qv}=\frac{\sqrt{1-q^2+(\tilde{b}\cos\rho-q\sin\rho)^2}}{\sin\rho-q\tilde{b}\cos\rho},$$

$$\mathrm{T}\equiv\frac{2\pi}{|w|}, \tag{3.43}$$

$$r=w\frac{q+\tilde{b}\tan\rho}{q\tilde{b}-\tan\rho}. \tag{3.44}$$

Here the independent parameters are ρ and $\tilde{b}$ associated, respectively, with two conserved charges – J_2 and p (see (3.33), (3.38)). The expression for ϕ_1 has indeed the same form as for the $J_2=0$, $q=0$ case, i.e. it depends on ρ (or J_2) and q only via $\tilde{b}$ in (3.29). Then a natural definition of the world-sheet momentum corresponding to ϕ_1 is

$$\mathrm{p}=\int d\theta\,\partial_t\phi_1\partial_\theta\phi_1=\int d\theta\,\partial_\theta\phi_1=\Delta\phi_1, \tag{3.45}$$

where we have taken into account that $\partial_t\phi_1(t,\theta)=1$.[12] The same conclusion is indeed reached in the uniform light-cone gauge where one has [23] $x_-=\varphi-t$, $x_+=(1-a)t+a\varphi=\tau$,

[11] This form of the solution is also useful for understanding its qualitative features. In particular, we see that the three parameters $\tilde{b}$, w and r each control a different type of behaviour. $\tilde{b}=\cot\frac{\Delta\phi_1}{2}\in(-\infty,\infty)$ measures the distance between the end points of the string, while $w\in(-1,1)$ measures the angular velocity in the ϕ_2 direction. The string also winds in the ϕ_2 direction and the size of these windings is controlled by $r\in(-\infty,\infty)$. From the expressions for $\tilde{b}$, w and r in (3.29), (3.43), (3.44) it is clear that the NS–NS flux does not introduce any new qualitative behaviours, i.e. they are all present for $q=0$. However, for fixed q (i.e. for a given string background) the solution is parametrised by only two independent parameters (for example (v,ρ) or $(\tilde{b},\rho)$) and hence only certain combinations of these three behaviours are allowed. As we let q vary the NS–NS flux can support certain configurations that would not otherwise be obtainable.

[12] Let us note also that general, given a rigid moving-wave soliton described by some profile function $\varphi=\varphi(x-vt)$ one may define the momentum as $\mathrm{p}=\int dx\,p_\varphi\varphi'$ where p_φ is the momentum density corresponding to f. For the S^2 giant magnon described in the light-cone gauge with constant J_1-density, this leads to $\mathrm{p}=2\arccos v$ [24]. One can see that the relation $\cos\frac{\mathrm{p}}{2}=\mathrm{v}$ between p and the soliton centre of mass velocity v generalises also to $J_2\neq0$ and $q\neq0$ cases (v $=v$

$p_+ = (1-a)p_\varphi - ap_t = 1$, $p_- = p_\varphi + p_t$. Here a is a gauge parameter (we ignore winding in the φ direction as we are interested in the decompactification limit $J_1 \to \infty$). The Virasoro condition (which is unchanged by the presence of the WZ term $\sim q$) then implies $\acute{x}_- + p_i \acute{x}^i = 0$. In the present case φ is to be identified with ϕ_1 (see [1]) and x^i stand for all other "transverse" coordinates. Thus the world-sheet momentum is $p_{\rm ws} \equiv -\int d\sigma\, p_i \acute{x}^i = \int d\sigma\, \acute{x}_- = \Delta\varphi = \Delta\phi_1 = \mathrm{p}$.[13]

Using (3.40) in (3.39) we arrive at the following $q \neq 0$ generalisation of the dyonic magnon dispersion relation[14]

$$E - J_1 = \sqrt{(J_2 - q\mathrm{h}\mathrm{p})^2 + 4\mathrm{h}^2\big(1-q^2\big)\sin^2\frac{\mathrm{p}}{2}}. \tag{3.46}$$

It is worth noting that, as in the $q = 0$ case (see Eq. (3.9)), the generalisation of this dispersion relation to the case when the string also moves along an S^1 in the torus part of the background is simply given by replacing $E \to \sqrt{E^2 - P^2}$ where P is the (large) momentum in S^1. Again the reason for this is the formal Lorentz invariance in the $\mathbb{R}_t \times S^1_\psi$ subspace in the decompactification limit.

Finally, let us derive the quantization condition for J_2 (a similar argument for the $q = 0$ case appeared in [7,8]). As one can see from (3.28) or (3.42) the giant magnon motion is time-periodic in the ϕ_2 direction with period T (3.43) assuming that the shift of t is compensated by a shift of x so that $\mathcal{X} + q\mathcal{T}$ and thus θ stays unchanged. In fact, the solution is explicitly periodic in $x_+ = \tau$ in the light-cone gauge discussed above, where $x_+ = (1-a)t + a\phi_1 = t + ax_- = \tau$ ($x_- = f(\theta)$, $\phi_2 = w\tau + g(\theta)$, see (3.41), (3.42)). The changes over the period $\delta t = \mathrm{T}$ are $\delta\theta = 0$, $\delta\phi_2 = 2\pi$, $\delta\phi_1 = \delta t = \mathrm{T}$ so that $\delta x_- = \delta\phi_1 - \delta t = 0$, $\delta x_+ = \mathrm{T}$.

This periodicity implies that there is an associated action variable, which should take integer values upon semiclassical quantization. Indeed, in general, for an integrable Hamiltonian system one can define action variables $I_s = \frac{1}{2\pi}\int_{\gamma_s} p_i\, \mathrm{d}q^i$ where the γ_s form a basis of Liouville torus cycles. The Bohr–Sommerfeld condition then implies that I_s should take integer values in the quantum theory. In the present case we can obtain the action variable I associated to the above cycle in phase space from[15]

$$2\pi I = S - \mathrm{T}\frac{\partial S}{\partial \mathrm{T}}\bigg|_{\mathrm{p}}. \tag{3.47}$$

Here $S = S(\mathrm{T}, \mathrm{p})$ is the light-cone gauge string action computed over one period T on the giant magnon solution (we assume that the parameters ρ and $\tilde{b}$ are expressed in terms of T in (3.43)

---

in (3.1) when $\rho = 0$). From (3.27), (3.28) the string centre of mass coordinates are $z_i = \lim_{\kappa\to\infty}\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{2\pi} Z_i(t, \kappa\sigma)$, i.e. $z_1 = \frac{\tilde{b}}{\sqrt{1+\tilde{b}^2}} e^{it}$, $z_2 = 0$, i.e. they describe a motion along a circle in the (X_1, X_2) plane with linear (tangent) velocity given by (cf. (3.33)) $\mathrm{v} = \frac{\tilde{b}}{\sqrt{1+\tilde{b}^2}} = \cos\frac{\Delta\phi_1}{2}$.

13 Thus the relation between world-sheet momentum and $\Delta\phi_1$ does not depend on the gauge parameter a; this was observed for $q = 0$ in [24] and agrees also with the near-BMN expansion for $q \neq 0$ in [1].

14 For completeness, one should also check that the "off-diagonal" components of the $SU(2)$ charges (2.17) vanish on this solution and indeed this is the case. We can also construct these "off-diagonal" charges by considering the corresponding Noether currents following from the local action (2.21). For these charges to be well-defined, i.e. for the spatial component of the current to go to zero as $x \to \pm\infty$ we again find that we should fix $\mathrm{c} = 1$. Furthermore, these charges also vanish on the solution, i.e. there is no additional contribution from the non-trivial boundary conditions.

15 Note that $\int p_i\, \mathrm{d}q^i = \int_0^{\mathrm{T}} \mathrm{d}t \int_{-\infty}^{+\infty} \mathrm{d}x\, p_i \dot{q}^i = S + \mathrm{T}H = S - \mathrm{T}\frac{\partial S}{\partial \mathrm{T}}$ where we used the Hamilton–Jacobi equation $H = -\frac{\partial S(t)}{\partial t}$.

and p in (3.40), (3.33)). Since the string action is reparametrisation-invariant, its value is gauge-independent and so it can be evaluated, e.g., in the conformal gauge (even though the periodicity of the solution is not manifest in this gauge – ϕ_1 gets an additional shift $\sim$ T). Considering $w > 0$ we compute the action (2.21) (keeping c arbitrary and including the $-\partial^a t \partial_a t$ term) on the solution (3.30), (3.31) to find

$$S = 2\pi \mathrm{h}\left[-\frac{2(1-q^2)}{\tan\rho - q\tilde{b}} + \frac{1}{2}q(\mathrm{c}+1)\Delta\phi_1 - \frac{1}{4\pi}\mathrm{T}q(\mathrm{c}-1)\Delta\phi_2\right], \tag{3.48}$$

where T is given in (3.43), $\Delta\phi_1$ in (3.33), (3.40), and $\Delta\phi_2$ in (3.37) is divergent. Thus the action, like $E - J_1$ in (3.36), is finite only if $\mathrm{c} = 1$, once again supporting the choice of the boundary term made above in (2.35), (2.36). Eliminating ρ in favour of T or $w = \frac{2\pi}{\mathrm{T}}$ (recall we consider $w > 0$) using (3.43), i.e.

$$\tan^2\rho = \frac{1 - q^2 + \tilde{b}^2 - [\sqrt{(1-q^2)(1+\tilde{b}^2)(1-w^2)} - qw\tilde{b}]^2}{1-w^2}, \tag{3.49}$$

gives (here $\tilde{b} = \cot\frac{\mathrm{p}}{2}$ and $w = \frac{2\pi}{\mathrm{T}}$)

$$\mathrm{c} = 1: \quad \frac{S}{2\pi\mathrm{h}} = \frac{2(1-q^2)\sqrt{1-w^2}}{q\tilde{b}\sqrt{1-w^2} - (1 - q^2 + \tilde{b}^2 - [\sqrt{(1-q^2)(1+\tilde{b}^2)(1-w^2)} - qw\tilde{b}]^2)^{1/2}} + q\mathrm{p}. \tag{3.50}$$

Substituting into (3.47) we find that the action variable associated to the periodic motion in ϕ_2 is nothing but J_2 given in (3.38), i.e.

$$I = J_2. \tag{3.51}$$

Thus J_2 should be quantized, which is consistent with the near-BMN perturbation theory [1] where the dispersion relation is a limit of (3.46) with $J_2 = 1$, and with the bound-state analysis in Section 5.

4. Giant magnon in the Landau–Lifshitz limit

Let us now check (3.40), (3.46) by considering a particular large angular momentum limit (when both J_1 and J_2 are large) in which the string action reduces to a Landau–Lifshitz (LL) model in which there is a natural definition for the world-sheet momentum.

In the $q = 0$ case the LL model admits a well-known "spin wave" soliton solution which, in fact, may be interpreted as a limit of the giant magnon of the original string sigma model, and we shall find its generalisation to $q \neq 0$. In the LL model one can give a natural definition to the spatial 2-d momentum of the soliton and as we shall see it is consistent with (3.40), (3.45) and the resulting energy–momentum relation agrees with large J_2 expansion of (3.46).

4.1. Landau–Lifshitz model for $q \neq 0$

To derive the LL model from the string action on $\mathbb{R} \times S^3$ one introduces a collective coordinate to isolate the "fast" string motion associated to the large total angular momentum and obtains the effective action describing the remaining "slow" degrees of freedom [13–15]. Let us parametrise S^3 as

$$
\begin{aligned}
&Z_1 = X_1 + iX_2 = \sin\theta e^{i\phi_1} = U_1 e^{i\alpha}, \quad U_1 = \sin\theta e^{i\beta}, \\
&Z_2 = X_3 + iX_4 = \cos\theta e^{i\phi_2} = U_2 e^{i\alpha}, \quad U_2 = \cos\theta e^{-i\beta}, \\
&\quad \alpha = \frac{1}{2}(\phi_1 + \phi_2), \ \beta = \frac{1}{2}(\phi_1 - \phi_2), \ |U_1|^2 + |U_2|^2 = 1.
\end{aligned}
\tag{4.1}
$$

The angle α and the $\mathbb{C}\mathbf{P}^1$ coordinates U_1, U_2 correspond to the S^1 Hopf fibration of S^3. The conformal-gauge string Lagrangian is $\mathcal{L} = -\frac{1}{2}\partial_+ t\partial_- t + \frac{1}{2}\mathcal{L}_S$ where the S^3 part in (2.21) written in the above coordinates takes the form

$$
\begin{aligned}
\mathcal{L}_S = {} & \partial_+\theta\partial_-\theta + \partial_+\alpha\partial_-\alpha + \partial_+\beta\partial_-\beta - (1+q)\partial_+\alpha C_- - (1-q)\partial_-\alpha C_+ \\
& - q\mathrm{c}(\partial_+\alpha\partial_-\beta - \partial_+\beta\partial_-\alpha), \quad C_\pm = \cos 2\theta\,\partial_\pm\beta.
\end{aligned}
\tag{4.2}
$$

With $t = \kappa\tau$ the Virasoro constraints are

$$
(\partial_\pm\alpha)^2 - 2\partial_\pm\alpha C_\pm + (\partial_\pm\theta)^2 + (\partial_\pm\beta)^2 = \kappa^2. \tag{4.3}
$$

Introducing $n_i = U^\dagger\sigma_i U$ or explicitly

$$
\vec{n} = (\sin 2\theta\cos 2\beta, \sin 2\theta\sin 2\beta, \cos 2\theta), \qquad \vec{n}^2 = 1, \tag{4.4}
$$

$$
\begin{aligned}
&\partial_+ C_- - \partial_- C_+ = -\frac{1}{2}\varepsilon_{ijk} n_i\partial_+ n_j\partial_- n_k, \\
&\frac{1}{4}\partial_+\vec{n}\cdot\partial_-\vec{n} = \partial_+\theta\partial_-\theta + \partial_+\beta\partial_-\beta - C_+ C_-,
\end{aligned}
\tag{4.5}
$$

we may rewrite the Lagrangian in (2.21) and the Virasoro constraints as

$$
\begin{aligned}
\mathcal{L}_S = {} & \frac{1}{4}\partial_+ n\cdot\partial_- n + (\partial_+\alpha - C_+)(\partial_-\alpha - C_-) - q(\partial_+\alpha C_- - \partial_-\alpha C_+) \\
& - q\mathrm{c}(\partial_+\alpha\partial_-\beta - \partial_+\beta\partial_-\alpha),
\end{aligned}
\tag{4.6}
$$

$$
\partial_\pm\alpha - C_\pm = \kappa\sqrt{1 - \frac{(\partial_\pm n)^2}{4\kappa^2}}. \tag{4.7}
$$

Let us now take the large total angular momentum limit directly in the action (as in [14]) using the Virasoro constraints to eliminate α.[16] Introducing $u = \alpha - t$ and expanding in large κ (which corresponds to large angular momentum limit with both J_1 and J_2 being large[17]) we find, after solving for u using the Virasoro constraints $\partial_\pm u = C_\pm + O(\kappa^{-1})$,[18]

$$
\mathcal{L}_S = \frac{1}{4}\partial_+ n\cdot\partial_- n - 2\kappa C_\tau + 2q\kappa C_\sigma + 2q\mathrm{c}\kappa\partial_\sigma\beta + O\big(\kappa^{-1}\big). \tag{4.8}
$$

Finally, using the equation of motion $\partial_\tau n_i = q\partial_\sigma n_i + O(\kappa^{-1})$ we arrive at the following $q \neq 0$ generalisation of the Landau–Lifshitz action

[16] Here we take the limit directly in the action rather than the equations of motion in order to determine the contribution to the LL action from the total derivative in the WZ term.

[17] This one can see from the large κ expansion of $J_\alpha = 2(J_1 + J_2) = \kappa + \int \frac{d\sigma}{2\pi} q[\mathrm{c} + \cos(2\theta)]\acute{\beta} + O(\kappa^{-1})$.

[18] We dropped the total derivative term $2\kappa\partial_\tau u$ and the constant term κ^2 which do not depend on q.

$$S_{\mathrm{LL}} = -\mathrm{h}\int \mathrm{d}t\,\mathrm{d}x\left[C_t - qC_x + \frac{1}{8}\left(1-q^2\right)(\partial_x n_i)^2 - q\mathrm{c}\partial_x\beta\right], \tag{4.9}$$

where $t = \kappa\tau$, $x = \kappa\sigma$ and $C_a = \cos 2\theta\ \partial_a\beta$.[19]

The LL model action (4.9) is invariant under translations of (t, x) and $SO(3)$ rotations of n_i. The former give two conserved charges – 2-d energy and 2-d momentum of the "slow" variables (which are no longer fixed by the Virasoro constraints)

$$E_{\mathrm{LL}} = \mathrm{h}\int \mathrm{d}x\left(-q(\cos 2\theta + \mathrm{c})\partial_x\beta + \frac{1}{2}\left(1-q^2\right)\left[(\partial_x\theta)^2 + \sin^2 2\theta(\partial_x\beta)^2\right]\right), \tag{4.10}$$

$$P_{\mathrm{LL}} = -\frac{\mathrm{h}}{2}\int \mathrm{d}x\,\frac{\partial\mathcal{L}_S}{\partial(\partial_t\beta)}\partial_x\beta = -\mathrm{h}\int \mathrm{d}x\,\cos 2\theta\,\partial_x\beta. \tag{4.11}$$

Then

$$E_{\mathrm{LL}} + qP_{\mathrm{LL}} = \mathrm{h}\int \mathrm{d}x\left(-q\mathrm{c}\partial_x\beta + \frac{1}{2}\left(1-q^2\right)\left[(\partial_x\theta)^2 + \sin^2 2\theta(\partial_x\beta)^2\right]\right). \tag{4.12}$$

Before discussing the LL model counterpart of the giant magnon solution let us consider the corresponding LL limit of the rigid circular string solution of Section 2.3. The solution in (2.47), (2.48) may be written as

$$\cos\theta = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{qm}{\sqrt{\kappa^2 - (1-q^2)m^2}}}, \qquad \alpha = \sqrt{\kappa^2 - \left(1-q^2\right)m^2}\,\tau,$$
$$\beta = m(\sigma + q\tau). \tag{4.13}$$

Taking the large κ limit we get, to leading order,

$$\cos\theta = \frac{1}{\sqrt{2}}, \qquad \alpha = \left[\kappa - \frac{(1-q^2)m^2}{2\kappa}\right]\tau, \qquad \beta = m(\sigma + q\tau), \tag{4.14}$$

which solve the LL equations of motion. The corresponding conserved charges (4.10), (4.11) are

$$E_{\mathrm{LL}} = 2\pi\mathrm{h}\left[-q\mathrm{c}m + \frac{1}{2}\left(1-q^2\right)\frac{m^2}{\kappa}\right], \qquad P_{\mathrm{LL}} = 0, \tag{4.15}$$

which is consistent with a large angular momentum expansion of the full string energy (2.52) (here $E_{\mathrm{LL}} = E - J_1$ and $J_1 = 2\pi\mathrm{h}\kappa$).

---

[19] In this procedure we use the Virasoro constraints in the action which in general may not necessarily lead to a correct result but in the present case indeed gives the same expression for the LL action as the systematic procedure based on uniform gauge fixing and large κ expansion developed in [14]. The same conclusion is also easily reached for $q \neq 0$ by taking the same limit directly at the level of string equations of motion:

$$\partial_+ n_i(\partial_-\alpha - C_-) + \partial_- n_i(\partial_+\alpha - C_+) - \varepsilon_{ijk}n_j\partial_+\partial_- n_k + q(\partial_+\alpha\partial_- n_i - \partial_-\alpha\partial_+ n_i) + O\left(\kappa^{-1}\right) = 0.$$

Using the expansion of the Virasoro constraints (4.7) to eliminate α gives

$$2\kappa(\partial_\tau - q\partial_\sigma)n_i + \varepsilon_{ijk}n_j\left(\partial_\sigma^2 - \partial_\tau^2\right)n_k + q(C_+\partial_- n_i - C_-\partial_+ n_i) + O\left(\kappa^{-1}\right) = 0.$$

For $q \neq 0$ the time derivatives of n_i are not suppressed but go as $\partial_\tau n_i = q\partial_\sigma n_i + O(\kappa^{-1})$. Eliminating them recursively from the above equation gives $(\partial_\tau - q\partial_\sigma)n_i = -\frac{1}{2\kappa}(1-q^2)\varepsilon_{ijk}n_j\partial_\sigma^2 n_k + O(\kappa^{-2})$ which follows from the action (4.9).

4.2. Landau–Lifshitz limit of the dyonic giant magnon solution

When taking the Landau–Lifshitz or large κ limit we required that the derivative $\partial_\sigma n_i$ stays finite. However, since the giant magnon solution is itself a large κ solution depending only on $(t, x) = (\kappa\tau, \kappa\sigma)$, in this case $\partial_\sigma n_i \sim O(\kappa)$. Therefore we also need to take an appropriate limit of the parameters in (3.27)–(3.29) to obtain the corresponding solution of the LL model. We have from (3.30), (3.31)

$$\begin{aligned}\partial_\sigma \cos\theta &= -\frac{\kappa\cos\rho}{\sqrt{1+\tilde{b}^2}}\frac{1-qv}{\sqrt{1-v^2}}\frac{\tanh(\cos\rho(\mathcal{X}+q\mathcal{T}))}{\cosh(\cos\rho(\mathcal{X}+q\mathcal{T}))},\\ \partial_\sigma\beta &= \kappa\cos\rho\frac{1-qv}{\tilde{b}\sqrt{1-v^2}}\frac{\cos^2(\arctan(\tilde{b}^{-1}\tanh[\cos\rho(\mathcal{X}+q\mathcal{T})]))}{\cosh(\cos\rho(\mathcal{X}+q\mathcal{T}))}\\ &\quad+\frac{\kappa}{2}\left(q-\sin\rho\frac{q-v}{\sqrt{1-v^2}}\right),\end{aligned} \tag{4.16}$$

so that to take the LL limit we need to assume that in the large κ limit

$$\sin\rho \sim 1 + O\left(\kappa^{-2}\right), \qquad v \sim O\left(\kappa^{-1}\right). \tag{4.17}$$

In this limit we have (cf. (3.2), (3.29))

$$\tan\rho \sim \kappa \gg 1, \qquad b = \frac{v\sec\rho}{\sqrt{1-v^2}} = \text{fixed}, \qquad \tilde{b} = b + O\left(\kappa^{-1}\right). \tag{4.18}$$

Under this assumption the conserved charges (3.34), (3.35) of the $q \neq 0$ dyonic giant magnon take the form (c = 1)

$$\begin{aligned}&E - J_1 = \frac{2\mathrm{h}\kappa}{1+b^2} + O\left(\kappa^0\right), \qquad J_2 = \frac{2\mathrm{h}\kappa}{1+b^2} + O\left(\kappa^0\right),\\ &\Delta\phi_1 = 2\arctan b^{-1} + O\left(\kappa^{-1}\right),\\ &\frac{E-J_1}{J_2} = 1 - q\frac{(1+b^2)\arctan b^{-1}}{\kappa} + O\left(\kappa^{-2}\right).\end{aligned} \tag{4.19}$$

Thus in the Landau–Lifshitz limit $E - J_1$ and J_2 diverge with their ratio staying finite. Eliminating b and κ from the above expressions we reproduce the large J_2 expansion of (3.46)

$$E - J_1 = J_2 - q\mathrm{h}\Delta\phi_1 + O\left(J_2^{-1}\right). \tag{4.20}$$

To construct the corresponding LL solution let us first consider the $q = 0$ case. Expanding the $q = 0$ giant magnon (3.1) at large κ we get[20]

$$\begin{aligned}2\alpha &= -b\sigma + 2\kappa\tau + \frac{b^2-1}{2\kappa}\tau + \arctan\left[b^{-1}\tanh\left(\sigma - \frac{b}{\kappa}\tau\right)\right],\\ 2\beta &= b\sigma - \frac{b^2-1}{2\kappa}\tau + \arctan\left[b^{-1}\tanh\left(\sigma - \frac{b}{\kappa}\tau\right)\right],\\ \cos\theta &= \frac{\operatorname{sech}(\sigma - \frac{b}{\kappa}\tau)}{\sqrt{1+b^2}}.\end{aligned} \tag{4.21}$$

[20] Note also that the Virasoro constraints take the expected form $\partial_\pm\alpha - C_\pm = \kappa - \frac{1}{2\kappa}\operatorname{sech}^2(\sigma - \frac{b}{\kappa}\tau)$.

These β and θ indeed solve the $q=0$ LL equations: they describe the known "pulse" or "spin wave" LL soliton found in [16–18].[21] The corresponding conserved charges are

$$E_{\rm LL} = \frac{\rm h}{2}\int {\rm d}x\,\big(\theta'^2 + \sin^2 2\theta\beta'^2\big) = \frac{\rm h}{\kappa}, \tag{4.22}$$

$$P_{\rm LL} = {\rm h}\int {\rm d}x\,(1+\cos 2\theta)\beta' = 2{\rm h}\arctan b^{-1}, \tag{4.23}$$

$$J_\beta = {\rm h}\int {\rm d}x\,(1+\cos 2\theta) = \frac{4{\rm h}\kappa}{1+b^2}, \tag{4.24}$$

where we have subtracted the values for the ground state solution $\theta = \pi/2$, $\phi_1 = \kappa\tau$, $\phi_2 = 0$ to obtain finite expressions. Here J_β is the angular momentum corresponding to translations in β.

Comparing with (4.19) we see that in the Landau–Lifshitz limit we have

$$P_{\rm LL} = {\rm h}\Delta\phi_1, \qquad J_\beta = 2J_2. \tag{4.25}$$

This then supports the identification of $\Delta\phi_1$ with the spatial momentum p and leads to the following familiar dispersion relation for the LL soliton

$$E_{\rm LL} = \frac{2{\rm h}^2}{J_2}\sin^2\frac{\rm p}{2}, \qquad {\rm p} = {\rm h}^{-1}P_{\rm LL}, \tag{4.26}$$

which is also the leading term in the large J_2 expansion of the dyonic giant magnon energy in (3.8), $E - J_1 \to J_2 + E_{\rm LL}$.[22]

The generalisation of the relevant large κ expansion of the giant magnon solution to $q \neq 0$ is

$$\begin{aligned}
2\alpha &= -b(\sigma + q\tau) + 2\kappa\tau + \frac{(1-q^2)(b^2-1)}{2\kappa}\tau \\
&\quad + \arctan\left[b^{-1}\tanh\left(\sigma + q\tau - \frac{b}{\kappa}(1-q^2)\tau\right)\right], \\
2\beta &= b(\sigma + q\tau) - \frac{(1-q^2)(b^2-1)}{2\kappa}\tau + \arctan\left[b^{-1}\tanh\left(\sigma + q\tau - \frac{b}{\kappa}(1-q^2)\tau\right)\right], \\
\cos\theta &= \frac{{\rm sech}[\sigma + q\tau - \frac{b}{\kappa}(1-q^2)\tau]}{\sqrt{1+b^2}}.
\end{aligned} \tag{4.27}$$

These β and θ satisfy the $q \neq 0$ LL equations of motion for (4.9) while α solves the Virasoro constrains

$$\partial_\pm\alpha - C_\pm = \kappa - \frac{1}{2\kappa}(1-q^2){\rm sech}^2\left[\sigma + q\tau - \frac{b}{\kappa}(1-q^2)\tau\right]. \tag{4.28}$$

Note that one can also obtain the $q \neq 0$ solution for β and θ by applying the world-sheet coordinate transformation $\tilde\tau = \tau$, $\sigma \to \tilde\sigma = \sigma - q\tau$, $\partial_\sigma = \tilde\partial_\sigma$, $\partial_\tau = \tilde\partial_\tau - q\tilde\partial_\sigma$ after which the LL equations following from (4.9) take the standard form $\tilde\partial_\tau n_i = \frac{1}{2\kappa}(1-q^2)\varepsilon_{ijk}n_j\tilde\partial_\sigma^2 n_k$.

[21] This soliton is non-topological (i.e. it can be continuously deformed into the vacuum $\theta = \frac{\pi}{2}$). Upon semiclassical quantization [25,18] its $U(1)$ charge J_2 is quantized and the quantum soliton $J_2 = 1$ state may be identified with the elementary magnon state.

[22] In the $AdS_5 \times S^5$ case the leading term of the expansion is protected and thus it also agrees with the small h expansion of the dyonic giant magnon energy, matching the expression following from the coherent state expectation value of the one-loop ferromagnetic spin chain Hamiltonian.

The corresponding energy (4.10) that generalises (4.22) to the $q \neq 0$ case is found to be (taking $\mathrm{c} = 1$)

$$\begin{aligned} E_{\mathrm{LL}} &= \mathrm{h} \int \mathrm{d}x \left[-q(1+\cos 2\theta)\beta' + \frac{1}{2}\left(1-q^2\right)\left(\theta'^2 + \sin^2 2\theta \beta'^2\right) \right] \\ &= -2\mathrm{h}q \arctan b^{-1} + \left(1-q^2\right)\frac{\mathrm{h}}{\kappa}, \end{aligned} \tag{4.29}$$

while the expressions for P_{LL} and J_β remain the same as in (4.23) and (4.24). As a result, Eq. (4.25) is unchanged while we find the following generalisation of the LL soliton dispersion relation (4.26)

$$E_{\mathrm{LL}} = -q\mathrm{hp} + \frac{2\mathrm{h}^2(1-q^2)}{J_2}\sin^2\frac{\mathrm{p}}{2}, \qquad \mathrm{p} = \mathrm{h}^{-1}P_{\mathrm{LL}}. \tag{4.30}$$

This agrees with the large J_2 expansion of the giant magnon energy (3.46) found in Section 3 thus supporting the identification of the magnon momentum (3.40) made there.

5. Symmetry algebra of light-cone gauge S-matrix and exact dispersion relation

In this section we will go back to the discussion of the world-sheet S-matrix of the mixed-flux $AdS_3 \times S^3$ theory of [1,2]. We shall first review the symmetry algebra that underlies the light-cone gauge S-matrix. Then we shall suggest a modification of one of the conjectures in [2] to find that we can recover the semiclassical $q \neq 0$ dyonic giant magnon dispersion relation (3.46) derived in Section 3.3 by considering the bound states of the theory and taking an appropriate strong-coupling limit.

5.1. Symmetry algebra

As the type IIB supergravity backgrounds with NS–NS and R–R 3-form fluxes are related by S-duality, the space–time symmetry of our background should not depend on q. Indeed, the $AdS_3 \times S^3$ part of the world-sheet action can be described by the same supercoset geometry $[PSU(1,1|2) \times PSU(1,1|2)]/[SU(1,1) \times SU(2)]$ [26] with q appearing only as a parameter in the action [27].

For this reason it is not surprising that the symmetry algebra of the world-sheet S-matrix [1,2] describing scattering above the BMN string (which should be a subalgebra of the supercoset symmetry preserved by the BMN vacuum) should not depend on q. The dependence on q then enters through the form of its representation on states [2].

The relevant symmetry takes the form of a direct sum of two copies of an algebra with the central extensions identified.[23] The generators of a single copy of this algebra are: (i) two $U(1)$ generators $\mathfrak{R}$ and $\mathfrak{L}$; (ii) four supercharges $\mathfrak{Q}_{\pm\mp}$ and $\mathfrak{S}_{\pm\mp}$ ($\pm$ denote the charges under the $U(1) \times U(1)$ bosonic subalgebra); (iii) three central extension generators $\mathfrak{C}$, $\mathfrak{P}$ and $\mathfrak{K}$. Defining

$$\mathfrak{M} = \frac{1}{2}(\mathfrak{R} + \mathfrak{L}), \qquad \mathfrak{B} = \frac{1}{2}(\mathfrak{R} - \mathfrak{L}), \tag{5.1}$$

the non-vanishing (anti-)commutation relations are given by

[23] The symmetry algebra here is the same as in the case of the S-matrix of the Pohlmeyer-reduced theory corresponding to the $AdS_3 \times S^3$ superstring [28,29].

$$[\mathfrak{B},\mathfrak{Q}_{\pm\mp}]=\pm i\mathfrak{Q}_{\pm\mp},\qquad [\mathfrak{B},\mathfrak{S}_{\pm\mp}]=\pm i\mathfrak{S}_{\pm\mp},$$
$$\{\mathfrak{Q}_{\pm\mp},\mathfrak{Q}_{\mp\pm}\}=\mathfrak{P},\qquad \{\mathfrak{S}_{\pm\mp},\mathfrak{S}_{\mp\pm}\}=\mathfrak{K},\qquad \{\mathfrak{Q}_{\pm\mp},\mathfrak{S}_{\mp\pm}\}=\pm i\mathfrak{M}+\mathfrak{C}. \tag{5.2}$$

These are consistent with the following set of reality conditions

$$\mathfrak{B}^\dagger=-\mathfrak{B},\qquad \mathfrak{Q}^\dagger_{\pm\mp}=\mathfrak{S}_{\mp\pm},\qquad \mathfrak{M}^\dagger=-\mathfrak{M},\qquad \mathfrak{P}^\dagger=\mathfrak{K},\qquad \mathfrak{C}^\dagger=\mathfrak{C}. \tag{5.3}$$

This superalgebra is a centrally-extended semi-direct sum of $\mathfrak{u}(1)$ (generated by $\mathfrak{B}$) with two copies of the superalgebra $\mathfrak{psu}(1|1)$, i.e.

$$\left[\mathfrak{u}(1)\Subset\mathfrak{psu}(1|1)^2\right]\ltimes\mathfrak{u}(1)\ltimes\mathbb{R}^3. \tag{5.4}$$

The central extensions are represented by the generators $\mathfrak{M}$ (corresponding to $\mathfrak{u}(1)$) and $\mathfrak{C}$, $\mathfrak{P}$ and $\mathfrak{K}$. There is only a single copy of the four central extensions in the symmetry of the full S-matrix

$$\left[\mathfrak{u}(1)\Subset\mathfrak{psu}(1|1)^2\right]^2\ltimes\mathfrak{u}(1)\ltimes\mathbb{R}^3. \tag{5.5}$$

The particular (reducible) representation of this symmetry algebra of interest to us here consists of one complex boson ϕ and one complex fermion ψ. The action of the $U(1)$ and fermionic generators are discussed in [2]; here we will just focus on the central extensions. These generators have the following action on the one-particle states

$$\{\mathfrak{M},\mathfrak{C},\mathfrak{P},\mathfrak{K}\}|\Phi_\pm\rangle=\left\{\pm\frac{i}{2}M_\pm,C_\pm,P_\pm,K_\pm\right\}|\Phi_\pm\rangle, \tag{5.6}$$

where $\Phi_\pm\in\{\phi_\pm,\psi_\pm\}$. These representation parameters should be real functions of the energy and momentum of the state. Furthermore, the closure of the algebra requires that these four parameters satisfy a constraint that is interpreted as the dispersion relation

$$C_\pm^2=\frac{M_\pm^2}{4}+P_\pm K_\pm. \tag{5.7}$$

The tree-level S-matrix was computed in [1] and from this result the leading-order expressions for the representation parameters in the near-BMN expansion ($\mathrm{h}\to\infty$) were written down in [2]:

$$M_\pm=1\pm qp,\qquad C_\pm=\frac{\varepsilon_\pm}{2},\qquad P_\pm=-\frac{i}{2}\sqrt{1-q^2}\,p,\qquad K_\pm=\frac{i}{2}\sqrt{1-q^2}\,p. \tag{5.8}$$

Here, the momentum p of a near-BMN excitation is related to the magnon momentum p of Sections 3 and 4 in the usual way, i.e. through a rescaling by the string tension h

$$p=\frac{\mathrm{p}}{\mathrm{h}}. \tag{5.9}$$

Substituting these near-BMN expressions (5.8) into (5.7) we reproduce the expected near-BMN dispersion relation [1]

$$\varepsilon_\pm=\sqrt{(1\pm qp)^2+\left(1-q^2\right)p^2}=\sqrt{1-q^2+(p\pm q)^2}. \tag{5.10}$$

Exact completions of $C_\pm$, $P_\pm$ and $K_\pm$ were then proposed in [2] based on various algebraic requirements and analogy with the pure R–R flux case:

$$C_\pm=\frac{\varepsilon_\pm}{2},\qquad P_\pm=\frac{\mathrm{h}}{2}\sqrt{1-q^2}\left(1-e^{i\mathrm{p}}\right),\qquad K_\pm=\frac{\mathrm{h}}{2}\sqrt{1-q^2}\left(1-e^{-i\mathrm{p}}\right). \tag{5.11}$$

Substituting these exact expression into the dispersion relation (5.7) we find

$$\varepsilon_{\pm} = \sqrt{M_{\pm}^2 + 4\mathrm{h}^2\left(1-q^2\right)\sin^2\frac{\mathrm{p}}{2}}. \tag{5.12}$$

An exact completion for $M_{\pm}$ conjectured in [2] was

$$M_{\pm} = 1 \pm 2q\mathrm{h}\sin\frac{\mathrm{p}}{2}. \tag{5.13}$$

However, it is now clear that this is not consistent with the semiclassical result (3.46). Instead, the expression that is consistent with both the near-BMN limit (5.10) and the semiclassical result (3.46) is simply

$$M_{\pm} = 1 \pm q\mathrm{hp}. \tag{5.14}$$

This alternative completion is not only compatible with the semiclassical result, but also has another advantage in that the construction of the bound-state dispersion relation is far more natural than in the case of (5.13), as we will explain below.

5.2. Zhukovsky variables

To discuss the bound-state dispersion relation we need to briefly describe the effect of the choice of (5.14) as oppose to (5.13) on the exact S-matrix. In [2] the S-matrix was written down in terms of the Zhukovsky variables $x_{\pm}^{\pm}$ (to be defined below), up to four overall phase factors. Various equations for these overall factors that follow from unitarity, braiding unitarity and crossing symmetry were listed there. Here we will not address the issue of these factors, which is still an open question for the mixed-flux case, i.e. will leave them unfixed.[24]

The key point is that the S-matrix expressed as a function of the Zhukovsky variables $x_{\pm}^{\pm}$ maintains exactly the same form as in [2]. It is only the map from $x_{\pm}^{\pm}$ to the energy/momentum of the scattering states and their dispersion relation that are modified due to the change from (5.13) to (5.14). It is important to note that many of the key properties of the S-matrix, including the Yang–Baxter equation, unitarity, braiding unitarity and crossing symmetry are satisfied (so long as the overall factors satisfy the equations as written in [2]) without the use of either this map or the dispersion relation.

The map between the Zhukovsky variables and the energy/momentum corresponding to (5.12) with arbitrary $M_{\pm}$ is a straight-forward generalisation of the familiar one[25]:

$$\begin{aligned} &e^{i\mathrm{p}} = \frac{x_{\pm}^{+}}{x_{\pm}^{-}}, \qquad \varepsilon_{\pm} = \frac{\mathrm{h}\sqrt{1-q^2}}{2i}\left(x_{\pm}^{+} - \frac{1}{x_{\pm}^{+}} - x_{\pm}^{-} + \frac{1}{x_{\pm}^{-}}\right), \\ &x_{\pm}^{\pm} = r_{\pm}e^{\pm i\frac{\mathrm{p}}{2}}, \qquad r_{\pm} = \frac{\varepsilon_{\pm} + M_{\pm}}{2\mathrm{h}\sqrt{1-q^2}\sin\frac{\mathrm{p}}{2}} = \frac{2\mathrm{h}\sqrt{1-q^2}\sin\frac{\mathrm{p}}{2}}{\varepsilon_{\pm} - M_{\pm}}, \end{aligned} \tag{5.15}$$

with the dispersion relation (5.12) expressed as

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[24] For the pure R–R case there have been a number of works studying these overall factors [30–33], leading to a conjecture in [4]. In [2] there was some discussion of the strong-coupling limit of these factors in the mixed-flux case. However, this discussion is likely to need modification in light of the results of this paper, and furthermore, the recipe presented there cannot be extend beyond this strong-coupling limit, due to inconsistencies with unitarity [34]. We thank R. Roiban for drawing this last point to our attention.

[25] Here the definitions of $x_{\pm}^{\pm}$ have a natural periodic extension of the region $\mathrm{p} \in (0, \pi)$ to the whole line, which is consistent with the semiclassical identification of p with the angular separation of the dyonic giant magnon string end-points.

$$x^+_\pm + \frac{1}{x^+_\pm} - x^-_\pm - \frac{1}{x^-_\pm} = \frac{2iM_\pm}{\mathrm{h}\sqrt{1-q^2}}. \tag{5.16}$$

As discussed above, the $M_\pm$ is unconstrained by the algebra, and hence could be an arbitrary function of the momentum provided it has the required near-BMN and strong-coupling limits. The choice of (5.14) corresponds to

$$M_\pm = 1 \pm q\mathrm{hp} = 1 \mp iq\mathrm{h}\log\frac{x^+_\pm}{x^-_\pm}. \tag{5.17}$$

Substituting (5.17) for $M_\pm$ in (5.16) we get

$$\left[\sqrt{1-q^2}\left(x^+_\pm + \frac{1}{x^+_\pm}\right) \mp 2q\log x^+_\pm\right] - \left[\sqrt{1-q^2}\left(x^-_\pm + \frac{1}{x^-_\pm}\right) \mp 2q\log x^-_\pm\right] = \frac{2i}{\mathrm{h}}. \tag{5.18}$$

It follows from this representation that we can define a "generalised" Zhukovsky map

$$\sqrt{1-q^2}\left(x_\pm + \frac{1}{x_\pm}\right) \mp 2q\log x_\pm = u, \qquad x^\pm_\pm = x_\pm\left(u \pm \frac{i}{\mathrm{h}}\right), \tag{5.19}$$

that "solves" the dispersion relation. However, it is apparent that the analytic structure of the inverse, $x_\pm(u)$, of (5.19) will be considerably more complicated than in the case of the pure R–R flux ($q = 0$), and indeed it is not clear whether this set of variables is the most illuminating for studying the complex structure of the spectral curve (5.18).

5.3. Bound-state dispersion relation

To construct the bound-state dispersion relation we first need to know the position of the poles of the S-matrix. Technically, for this we should also have the exact form of the four phase factors mentioned above. However, at the position of a pole corresponding to a bound state we expect that the residue of the S-matrix given in [2] should project onto a short representation of the symmetry algebra (i.e. a boson and a fermion). From the form of the S-matrix it is then clear that candidate positions for poles (and the corresponding short bound-state representations) include[26]

$$\begin{aligned}
&\text{(i):} \quad x^+_\pm = x'^-_\pm, \qquad \left\{|\phi_\pm\phi'_\pm\rangle, |\phi_\pm\psi'_\pm\rangle + \varphi_\pm\varphi'_\pm\frac{\eta_\pm}{\eta'_\pm}|\psi_\pm\phi'_\pm\rangle\right\},\\
&\text{(ii):} \quad x^-_\pm = x'^+_\pm, \qquad \left\{|\psi_\pm\psi'_\pm\rangle, |\phi_\pm\psi'_\pm\rangle - \varphi_\pm\varphi'_\pm\frac{\eta_\pm}{\eta'_\pm}|\psi_\pm\phi'_\pm\rangle\right\},
\end{aligned} \tag{5.20}$$

where

$$\varphi_\pm = \sqrt[4]{\frac{x^+_\pm}{x^-_\pm}}, \qquad \eta_\pm = \sqrt{i\left(x^-_\pm - x^+_\pm\right)}. \tag{5.21}$$

On physical grounds we would expect that the bound states should form in the sector associated to the 3-sphere and indeed this is what happens in the $q = 0$ case of pure R–R flux [35]. As the field $\phi_\pm$ is associated to the 3-sphere we expect there to be a pole corresponding to case (i) in (5.20), and not to case (ii).

[26] Here the unprimed and primed variables correspond to the two incoming particles in the scattering process.

The bound-state energy and momentum should be given by the sum of those of the two constituent states

$$E^{(2)}_{\pm} = \varepsilon_{\pm} + \varepsilon'_{\pm}, \qquad \mathrm{p}^{(2)} = \mathrm{p} + \mathrm{p}'. \tag{5.22}$$

From (5.15) it immediately follows that

$$e^{i\mathrm{p}^{(2)}} = \frac{x'^{+}_{\pm}}{x^{-}_{\pm}}, \qquad E^{(2)}_{\pm} = \frac{\mathrm{h}\sqrt{1-q^2}}{2i}\left(x'^{+}_{\pm} - \frac{1}{x'^{+}_{\pm}} - x^{-}_{\pm} + \frac{1}{x^{-}_{\pm}}\right). \tag{5.23}$$

Therefore, we can interpret the Zhukovsky variables for the bound state as

$$x^{(2)+}_{\pm} = x'^{+}_{\pm}, \qquad x^{(2)-}_{\pm} = x^{-}_{\pm}. \tag{5.24}$$

Denoting $x^{+}_{\pm} = x'^{-}_{\pm} = \mathrm{x}_{\pm}$ at the position of the pole, the dispersion relations for the two constituent particles of the bound state are then

$$\begin{aligned} &\mathrm{x}_{\pm} + \frac{1}{\mathrm{x}_{\pm}} - x^{(2)-}_{\pm} - \frac{1}{x^{(2)-}_{\pm}} = \frac{2iM_{\pm}}{\mathrm{h}\sqrt{1-q^2}}, \\ &x^{(2)+}_{\pm} + \frac{1}{x^{(2)+}_{\pm}} - \mathrm{x}_{\pm} - \frac{1}{\mathrm{x}_{\pm}} = \frac{2iM'_{\pm}}{\mathrm{h}\sqrt{1-q^2}}. \end{aligned} \tag{5.25}$$

Summing these up we find the dispersion relation for the bound state

$$x^{(2)+}_{\pm} + \frac{1}{x^{(2)+}_{\pm}} - x^{(2)-}_{\pm} - \frac{1}{x^{(2)-}_{\pm}} = \frac{2iM^{(2)}_{\pm}}{\mathrm{h}\sqrt{1-q^2}}, \tag{5.26}$$

where

$$M^{(2)}_{\pm} = M_{\pm} + M'_{\pm}. \tag{5.27}$$

We thus see that the eigenvalue $M_{\pm}$ of the central generator $\mathfrak{M}$ is additive when acting on the two (and higher) particle states. Indeed, this follows immediately from the fact that its coproduct is the standard one [2,35,1,36] (i.e. it is just given by the usual Leibniz action).

It follows from the definition of $M_{\pm}$ in (5.17) that the value of $\mathfrak{M}$ acting on the bound state is

$$M^{(2)}_{\pm} = 2 \pm q\mathrm{h}\mathrm{p}^{(2)} = 2 \mp iq\mathrm{h}\log\frac{x^{(2)+}_{\pm}}{x^{(2)-}_{\pm}}. \tag{5.28}$$

It is crucial to note that the simplicity found here, in particular, the fact that $\mathrm{x}^{\pm}$ drop out without any additional work, is a direct consequence of the fact that $M_{\pm}$ in (5.14), (5.17) are *linear* functions of p. Furthermore, it is clear that this procedure will iterate, giving a tower of bound states with

$$M^{(N)}_{\pm} = N \pm q\mathrm{h}\mathrm{p}^{(N)} = N \mp iq\mathrm{h}\log\frac{x^{(N)+}_{\pm}}{x^{(N)-}_{\pm}}. \tag{5.29}$$

The resulting dispersion relation for the N-particle bound state is then

$$E^{(N)}_{\pm} = \sqrt{\left(N \pm q\mathrm{h}\mathrm{p}^{(N)}\right)^2 + 4\mathrm{h}^2\left(1-q^2\right)\sin^2\frac{\mathrm{p}^{(N)}}{2}}. \tag{5.30}$$

This agrees with the semiclassical result (3.46) after quantizing the angular momentum: $J_2 = N$.

It is worth noting that this agreement requires the bound-state momentum to satisfy the bound $|\mathrm{p}^{(N)}| \leqslant \pi$ (at least in the semiclassical $\mathrm{h} \to \infty$ limit). This is implied by the identification of this momentum with that of the semiclassical dyonic giant magnon, which in turn is given by the separation angle of the string end-points. This suggests that any momentum p should be understood as defined modulo 2π (and thus may be taken to lie in the range $|\mathrm{p}| \leqslant \pi$). Then the momentum conservation should also be considered modulo 2π.[27] The consequences of this rather unusual dispersion relation and its interpretation are important topics for further study.

6. Concluding remarks

In this paper we have supplemented the information provided by the perturbative near-BMN expansion [1] and the light-cone symmetry algebra [2] with the construction of the semiclassical dyonic giant magnon solution in $AdS_3 \times S^3 \times T^4$ string theory with mixed flux to propose the exact form of the corresponding dispersion relation. We have seen that the presence of the WZ term representing the NS–NS flux in the bosonic string action leads to subtleties associated to the proper choice of boundary terms and the definition of angular momenta, which become important for non-trivial open-string solutions like the giant magnon.

We reviewed the symmetry algebra for the string light-cone gauge S-matrix and introduced a new set of Zhukovsky variables corresponding to the proposed dispersion relation. Analysing the resulting bound-state dispersion relation, we found that it has a simple structure (5.30) and agrees with the giant magnon dispersion relation (3.46). The implications of this new dispersion relation for the structure of the yet undetermined "phase factors" in the exact S-matrix [2] remain to be studied.

It would be interesting to provide further checks of the dispersion relation (1.5), (1.7). One possibility would be through a two-loop perturbative string computation like that done in the $AdS_5 \times S^5$ [37] and $AdS_2 \times S^2 \times T^6$ cases [38]. It appears, however, that the near-flat space limit [39] used in these papers is not sufficient to determine the q-dependence of the two-loop correction to the dispersion relation (e.g., a potential qp^3 term will not be seen in this limit). Therefore, to check (1.7) one would need to do the full near-BMN two-loop computation of the two-point function, which is yet to be performed in the $AdS_5 \times S^5$ case. One may also get additional information about the perturbative expansion of the dispersion relation and S-matrix using unitarity-based methods [40,34].

Another check of (1.5), (1.7) would be to confirm that the first semiclassical (one-loop) correction to the giant magnon energy (3.46) vanishes[28] as was shown in the case of the $AdS_5 \times S^5$ giant magnon in [41–43]. This should indeed be the case since (i) the one-loop corrections in the string and the corresponding Pohlmeyer reduced theory should match [44] (since the classical equations and thus the leading fluctuations near a classical solution are directly related) and (ii) the solution of the reduced theory corresponding to the giant magnon is essentially the same as in the $q = 0$ case up to a simple rescaling of the mass scale by $\sqrt{1-q^2}$ (see [2] and Appendix A below).

[27] Note that in the $q \to 1$ limit the $\sin^2 \frac{\mathrm{p}}{2}$ term drops out of the dispersion relation and it might appear somewhat unnatural to take this definition. However, in this limit the classical string solution remains well-defined. Furthermore, $e^{\frac{i\mathrm{p}}{2}}$ still appears in the definition of $x^{\pm}_{\pm}$ and thus the S-matrix. These both suggest that the momentum should continue to be defined modulo 2π.

[28] In semiclassical limit $J_2 \sim h\mathcal{J}_2$ and $\mathcal{J}_2$ and p are fixed while one expands in large h.

Finally, to use the dispersion relation (1.5), (1.7) and the corresponding S-matrix as a starting point for computing the string spectrum, it would be important to have a better understanding of the analytic structure of the complex spectral curve (5.18), in particular, identifying the corresponding uniformizing variables (the analogs of those introduced in the $AdS_5 \times S^5$ case in [45,46]).

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Appendix A. Relation to soliton of the Pohlmeyer reduced theory

Let us briefly describe the relation between the $q \neq 0$ generalisation of the giant magnon solution, found in Section 3.2, and the corresponding soliton of the Pohlmeyer reduction of $\mathbb{R} \times S^3$ string theory with $q \neq 0$, which is the complex sine-Gordon model with the mass parameter rescaled by $\sqrt{1-q^2}$ [1]. This generalises the relation between the soliton of the complex sine-Gordon model and the $q = 0$ dyonic giant magnon used in [8].

The Lagrangian density of the Pohlmeyer reduced theory and the relation of the string embedding coordinates X_m to the reduced variables are given by [1]

$$\begin{aligned} &\mathcal{L} = \partial_+\varphi\partial_-\varphi + \tan^2\varphi\,\partial_+\chi\,\partial_-\chi + \frac{1}{2}\kappa^2\big(1-q^2\big)\cos 2\varphi, \\ &\kappa^2\cos 2\varphi = \partial_+ X\cdot\partial_- X, \qquad \kappa^3\sin^2\varphi\,\partial_\pm\chi = \mp\frac{1}{2}\varepsilon_{mnkl}X^m\partial_+X^n\partial_-X^k\partial_\pm^2X^l. \end{aligned} \tag{A.1}$$

These can be written in terms of the $SU(2)$ current $\mathfrak{J}$ in (2.7) as follows:

$$\kappa^2\cos 2\varphi = -\frac{1}{2}\mathrm{tr}(\mathfrak{J}_+\mathfrak{J}_-), \qquad \kappa^3\sin^2\varphi\,\partial_\pm\chi = \pm\frac{1}{8}\mathrm{tr}\big([\mathfrak{J}_+,\mathfrak{J}_-]\partial_\pm\mathfrak{J}_\pm\big). \tag{A.2}$$

Substituting the expressions (3.27)–(3.31) for the $q \neq 0$ giant magnon solution into $\mathfrak{J}_\pm$ the corresponding reduced theory solution is found to be

$$\sin\varphi = \frac{\cos\rho}{\cosh[\cos\rho(\mathcal{X}+q\mathcal{T})]}, \qquad \chi = 2\sin\rho(\mathcal{T}+q\mathcal{X}), \tag{A.3}$$

where $\mathcal{T}$ and $\mathcal{X}$ were defined in (3.3). Then

$$\psi \equiv \sin\varphi\, e^{\frac{i}{2}\chi} = \frac{\cos\rho\exp[i\sin\rho(\mathcal{T}+q\mathcal{X})]}{\cosh[\cos\rho(\mathcal{X}+q\mathcal{T})]} \tag{A.4}$$

is recognised as the familiar complex sine-Gordon soliton solution.

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